

Hypocoercivity of Tempered Bouncy Particle Sampler for heavy-tailed targets

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Part II: Ideas and Proofs (Hypocoercivity for T-BPS)

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Recap: Tempered Bouncy Particle Sampler

Choose smooth $\sigma : \mathbb{R}^d \rightarrow [1, \infty)$ and define the Hamiltonian

$$H(x, v) = U(x) + \frac{|v|^2}{2\sigma^2(x)} + d \log \sigma(x).$$

$$\begin{aligned} X_t &= X_0 + \int_0^t V_s \, ds, \\ V_t &= V_0 + \int_0^t (R_{\nabla_x H(X_s, V_{s-})} V_{s-} - V_{s-}) \, dN_s^b + \int_0^t (\tilde{V}_s - V_{s-}) \, dN_s^r, \end{aligned} \quad (\text{T-BPS})$$

- As in BPS, between events: $dX_t = V_t \, dt$, $dV_t = 0$.
- Bounce rate: $\lambda_b(x, v) = \langle v, \nabla_x H(x, v) \rangle_+$ with jump $v \mapsto R_{\nabla_x H(x, v)} v$.
- Refresh rate λ_r : $\tilde{V} \sim \kappa_x = \mathcal{N}(0, \sigma^2(x) I_d) = O(\sigma(x))$ (position-dependent refresh distribution).

Recap: Main result

(A0) Weighted Poincaré inequality: $m\|g - \mu(g)\|_{L^2(\mu)}^2 \leq \int \sigma^2 |\nabla g|^2 d\mu$

(A1) Tempering bounds: $|\nabla \sigma|, \sigma|\nabla^2 \sigma|_F, \sigma|\nabla U|, \sigma^2|\nabla^2 U|_F \leq M < \infty$

(A2) Weighted curvature lower bound: $\sigma^2 \nabla^2 U \succeq -K I_d$

Theorem ([MN26])

Let $(\mathcal{P}_t)$ be the T -BPS semigroup. Under (A0)–(A2),

$$\|\mathcal{P}_t f\|_{L^2(\mu_\sigma)}^2 \leq 3e^{-\nu t} \|f\|_{L^2(\mu_\sigma)}^2, \quad f \in L_0^2(\mu_\sigma).$$

For $K_0 \lesssim d^3 M^2 m^{-1} K$, the explicit rate is

$$\nu^{-1} = \frac{3}{\lambda_r} \left[4 + \left(2\sqrt{K_0} + \frac{\lambda_r}{2\sqrt{m}} \right)^2 \right].$$

Section 1

Introduction to Hypocoercivity

Introduction to Hypocoercivity (I)

Consider the overdamped Langevin dynamics (OLD) with invariant distribution $\mu(\mathrm{d}x) = Z_U^{-1} e^{-U(x)} \mathrm{d}x$ on $\mathbb{R}^d$ given by the SDE:

$$\mathrm{d}Y_t = -\nabla U(Y_t) \mathrm{d}t + \sqrt{2} \mathrm{d}W_t, \quad t \geq 0.$$

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Note: This is a reversible process.

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- Aim: To study how quickly $\mathcal{S}_t f \rightarrow \mu(f)$. I.e. how quickly $\mathcal{S}_t f \rightarrow 0$ for $f \in L_0^2(\mu) \iff \mu(f) = 0$.
- Therefore, we look at $\frac{d}{dt} \|\mathcal{S}_t f\|_{L^2(\mu)}^2$ for f with $\mu(f) = 0$.

Introduction to Hypocoercivity (II)

- Using chain rule, for $f \in L_0^2(\mu)$, we get

$$-\frac{d}{dt} \|\mathcal{S}_t f\|_{L^2(\mu)}^2 = 2 \langle \mathcal{S}_t f, -\mathcal{L}_0 \mathcal{S}_t f \rangle_{L^2(\mu)},$$

where $\mathcal{L}_0 = -\nabla^* \nabla$ is the generator of the semigroup $(\mathcal{S}_t)_{t \geq 0}$ of the Overdamped Langevin dynamics (OLD).

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- Def.:** $\mathcal{L}_0$ is **coercive** if there exists $m > 0$ such that

$$\langle g, -\mathcal{L}_0 g \rangle_{L^2(\mu)} \geq m \|g\|_{L^2(\mu)}^2 \text{ for all } g \in \text{Dom}(\mathcal{L}_0) \text{ with } \mu(g) = 0.$$

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- Consequence of coercivity:

$$\frac{d}{dt} \|\mathcal{S}_t f\|_{L^2(\mu)}^2 \leq -2m \|\mathcal{S}_t f\|_{L^2(\mu)}^2 \xrightarrow{\text{Grönwall}} \|\mathcal{S}_t f\|_{L^2(\mu)}^2 \leq e^{-2mt} \|f\|_{L^2(\mu)}^2.$$

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- Coercivity of $\mathcal{L}_0 = -\nabla^* \nabla \Leftrightarrow$ Poincaré-inequality for μ .

$$\|\nabla f\|_{L^2(\mu)}^2 \geq m \|f\|_{L^2(\mu)}^2 \text{ for all } f \text{ differentiable with } \mu(f) = 0$$

Introduction to Hypocoercivity (III)

Next, consider the underdamped Langevin dynamics (ULD) with invariant distribution $\tilde{\mu} = \mu \otimes \kappa$ (where κ is the Gaussian on $\mathbb{R}^d$), given by the following SDE on $\mathbb{R}^d \times \mathbb{R}^d$.

$$dX_t = V_t dt, \quad dV_t = -\nabla_x U(X_t) dt - \lambda V_t dt + \sqrt{2\lambda} dW_t.$$

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The semigroup of the above has the generator

$$\tilde{\mathcal{L}} = \underbrace{v \cdot \nabla_x - \nabla U \cdot \nabla_v}_{\mathcal{L}_{\text{tr}} \text{ (transport part)}} + \lambda \underbrace{(-\nabla_v^* \nabla_v)}_{\mathcal{L}_v \text{ (OU part)}}$$

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This is **non-coercive**:

$$\langle g, -\tilde{\mathcal{L}}g \rangle_{L^2(\tilde{\mu})} = \underbrace{\langle g, -\mathcal{L}_{\text{tr}}g \rangle_{L^2(\tilde{\mu})}}_{=0 \text{ as } \mathcal{L}_{\text{tr}}^* = -\mathcal{L}_{\text{tr}}} + \lambda \langle g, -\mathcal{L}_v g \rangle_{L^2(\tilde{\mu})} = \lambda \|\nabla_v g\|_{L^2(\tilde{\mu})}^2.$$

and $\|\nabla_v g\|_{L^2(\tilde{\mu})}^2 \not\geq m \|g\|_{L^2(\tilde{\mu})}^2$ for any $m > 0$ and g independent of v variable.

Introduction to Hypocoercivity (IV)

Hypocoercivity: $\tilde{\mathcal{L}}$ may be coercive in a Lyapunov functional $\mathcal{V}$ that is equivalent to $\|\cdot\|_{L^2(\tilde{\mu})}^2$. I.e., we want

- ① **Equivalence:** $c\|f\|_{L^2(\tilde{\mu})}^2 \leq \mathcal{V}(f) \leq C\|f\|_{L^2(\tilde{\mu})}^2$.
- ② **Coercivity in $\mathcal{V}$:** Define functional $\mathcal{E}$ by $-\frac{d}{dt}\mathcal{V}(\mathcal{P}_t f) =: \mathcal{E}(\mathcal{P}_t f)$. If there exists $\nu > 0$ such that

$$\mathcal{E}(f) \geq \nu \mathcal{V}(f) \quad \text{for all } f \text{ with } \tilde{\mu}(f) = 0.$$

Using Grönwall, ② implies $\mathcal{V}(\mathcal{P}_t f) \leq e^{-\nu t} \mathcal{V}(f)$. Using ①, this gives

$$\|\mathcal{P}_t f\|_{L^2(\tilde{\mu})}^2 \leq \left(\frac{C}{c}\right) e^{-\nu t} \|f\|_{L^2(\tilde{\mu})}^2.$$

Note that the constant $C/c > 1$.

Introduction to Hypocoercivity (V)

For the ULD generator $\tilde{\mathcal{L}}$, the Lyapunov function needed for **hypocoercivity** is of the following form [DMS10; FLL26]:

$$\mathcal{V}(f) := \frac{1}{2} \|f\|_{L^2(\tilde{\mu})}^2 - \delta \langle f, \underbrace{(m - \mathcal{L}_0)^{-1} \Pi \mathcal{L}_{\text{tr}}^*}_{\mathcal{A}} f \rangle_{L^2(\tilde{\mu})} \quad \text{for } \delta > 0 \text{ sufficiently small}$$

and the velocity-averaging operator $\Pi f(x) = \kappa_x(f(x, \cdot))$.

Since $\mathcal{E}(\mathcal{P}_t f) := -\frac{d}{dt} \mathcal{V}(\mathcal{P}_t f)$, we have

$$\mathcal{E}(f) = \langle f, -\tilde{\mathcal{L}}f \rangle_{L^2(\tilde{\mu})} + \delta \langle f, \mathcal{A}\tilde{\mathcal{L}}f \rangle_{L^2(\tilde{\mu})} + \delta \langle \tilde{\mathcal{L}}f, \mathcal{A}f \rangle_{L^2(\tilde{\mu})} \quad \text{for } f \in \text{Dom}(\tilde{\mathcal{L}}).$$

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Ingredients for coercivity of $\tilde{\mathcal{L}}$ in $\mathcal{V}$:

- The ULD generator $\tilde{\mathcal{L}}$ has to be the **second-order lift** [Ebe+25, Def. 1] of the OLD generator $\mathcal{L}_0$:

$$\langle \tilde{\mathcal{L}}h, \tilde{\mathcal{L}}g \rangle_{L^2(\tilde{\mu})} = \langle h, -\mathcal{L}_0 g \rangle_{L^2(\mu)}, \quad \langle h, \tilde{\mathcal{L}}g \rangle_{L^2(\tilde{\mu})} = 0 \text{ for } h, g \in \text{Dom}(\mathcal{L}_0).$$

- $\mathcal{L}_0$ has to be coercive.

This gives exponential decay rates for the ULD semigroup (e.g. [FLL26]).

Section 2

Hypocoercivity of TBPS

Generator of T-BPS

Define the operators

$$\begin{aligned}\mathcal{L}_{\text{tr}} f(x, v) &= \langle v, \nabla_x f(x, v) \rangle + \langle v, \nabla_x H(x, v) \rangle_+ (f(x, R_{\nabla_x H(x, v)} v) - f(x, v)), \\ \mathcal{L}_v &= \Pi - I,\end{aligned}$$

with the velocity-averaging operator $\Pi f(x, v) := \int_{\mathbb{R}^d} f(x, w) \kappa_x(dw)$.

Let $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ denote the generator of $(\mathcal{P}_t)_{t \geq 0}$.

Theorem

- (I) μ_σ is an invariant distribution for $(\mathcal{P}_t)_{t \in \mathbb{R}_+}$.
- (II) For $f \in C_c^1(\mathbb{R}^d \times \mathbb{R}^d) \subseteq \text{Dom}(\mathcal{L})$, $\mathcal{L}f = \mathcal{L}_{\text{tr}}f + \lambda_r \mathcal{L}_v f$.
- (III) $C_c^1(\mathbb{R}^d \times \mathbb{R}^d)$ is a core for the generator $(\mathcal{L}, \text{Dom}(\mathcal{L}))$.

Hypo-coercivity for T-BPS

The generator of the T-BPS also lacks coercivity: $\langle g, -\mathcal{L}g \rangle_{L^2(\mu_\sigma)} = 0$ for g independent of the v variable.

Hence, we build a similar hypo-coercivity framework for the T-BPS:

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- The T-BPS generator $\mathcal{L}$ is a **second-order lift** of the **Tempered Langevin dynamics** (TLD) generator $\mathcal{L}_\sigma = -\nabla^* \sigma^2 \nabla$.

TLD process:

$$dY_t = (2\sigma(Y_t)\nabla\sigma(Y_t) - \sigma^2(Y_t)\nabla U(Y_t)) ds + \sqrt{2}\sigma(Y_t) dB_s$$

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- Coercivity of TLD generator $\mathcal{L}_\sigma$: want existence of $m > 0$ such that

$$\begin{aligned} \langle g, -\mathcal{L}_\sigma g \rangle_{L^2(\mu)} &\geq m \|g\|_{L^2(\mu)}^2 \quad \text{for all } g \in \text{Dom}(\mathcal{L}_\sigma) \text{ with } \mu(g) = 0 \\ \Leftrightarrow \int \sigma^2 |\nabla g|^2 d\mu &\geq m \|g\|_{L^2(\mu)}^2 \quad \text{The weighted Poincaré inequality!} \end{aligned}$$

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- We propose the Lyapunov functional

$$\mathcal{V}_\delta(f) := \frac{1}{2} \|f\|_{L^2(\mu_\sigma)}^2 - \delta \langle f, (m - \mathcal{L}_\sigma)^{-1} \Pi \mathcal{L}_{\text{tr}}^* f \rangle_{L^2(\mu_\sigma)} \quad \text{for } \delta > 0.$$

Coercivity in $\mathcal{V}_\delta$ and Hypocoercivity

Let $\mathcal{E}_\delta$ be a functional such that $-\frac{d}{dt}\mathcal{V}_\delta(\mathcal{P}_t f) = \mathcal{E}_\delta(\mathcal{P}_t f)$.

Lemma

Let Assumptions (A0)–(A2) hold. For δ sufficiently small,

- ① $\exists \nu > 0$ such that $\mathcal{E}_\delta(f) \geq \nu \mathcal{V}_\delta(f)$ for all $f \in \text{Dom}(\mathcal{L})$ with $\mu_\sigma(f) = 0$.
- ② $\exists c, C > 0$ such that

$$c\|f\|_{L^2(\mu_\sigma)}^2 \leq \mathcal{V}_\delta(f) \leq C\|f\|_{L^2(\mu_\sigma)}^2 \quad \text{for all } f \in L^2(\mu_\sigma).$$

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Using Grönwall on ①, we have $\mathcal{V}_\delta(\mathcal{P}_t f) \leq e^{-\nu t} \mathcal{V}_\delta(f)$.

Using ②, this gives the required result:

$$\|\mathcal{P}_t f\|_{L^2(\mu_\sigma)}^2 \leq (C/c)e^{-\nu t} \|f\|_{L^2(\mu_\sigma)}^2.$$

References

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