

Stability of storage processes with general release rates

Miha Brešar¹ Aleksandar Mijatović² Nikola Sandrić³

¹Chinese University of Hong Kong, Shenzhen

²University of Warwick

³University of Zagreb

9 November, 2025

Storage process (setup and goals)

Storage process X satisfies

$$X_t = x + A(t) - \int_0^t r(X_s) ds, \quad x \geq 0, t \geq 0.$$

Storage process (setup and goals)

Storage process X satisfies

$$X_t = x + A(t) - \int_0^t r(X_s) ds, \quad x \geq 0, \quad t \geq 0.$$

Input A : non-decreasing pure-jump Lévy process (subordinator).

Storage process (setup and goals)

Storage process X satisfies

$$X_t = x + A(t) - \int_0^t r(X_s) ds, \quad x \geq 0, \quad t \geq 0.$$

Input A : non-decreasing pure-jump Lévy process (subordinator).

The Lévy measure ν of A satisfies $\nu((-\infty, 0]) = 0$, and

$$\int_{[0, \infty)} (u \wedge 1) \nu(du) < \infty.$$

Storage process (setup and goals)

Storage process X satisfies

$$X_t = x + A(t) - \int_0^t r(X_s) ds, \quad x \geq 0, t \geq 0.$$

Input A : non-decreasing pure-jump Lévy process (subordinator).

The Lévy measure ν of A satisfies $\nu((-\infty, 0]) = 0$, and

$$\int_{[0, \infty)} (u \wedge 1) \nu(du) < \infty.$$

Release rate r : $[0, \infty) \rightarrow [0, \infty)$, positive iff $u > 0$, locally Lipschitz on compacts.

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Recurrence/ergodicity for non-decreasing inputs: [Yam05]

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Recurrence/ergodicity for non-decreasing inputs: [Yam05]

Modern advances:

Connections to infinite-server queues [SBM20; DFP25].

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Recurrence/ergodicity for non-decreasing inputs: [Yam05]

Modern advances:

Connections to infinite-server queues [SBM20; DFP25].

Explicit formulas in the when X is a shot-noise process, i.e., $r(x) = cx$ [BM21b; BKM19].

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Recurrence/ergodicity for non-decreasing inputs: [Yam05]

Modern advances:

Connections to infinite-server queues [SBM20; DFP25].

Explicit formulas in the when X is a shot-noise process, i.e., $r(x) = cx$ [BM21b; BKM19].

Extensions to affine $r(x) = b + cx$ and two-sided jumps $\nu((-\infty, 0]) > 0$ [BM21a].

Classical problem with rich history:

Existence/stationarity for compound Poisson inputs: [HR76; BRT82]

Exponential ergodicity for compound Poisson inputs: [MT93]

Recurrence/ergodicity for non-decreasing inputs: [Yam05]

Modern advances:

Connections to infinite-server queues [SBM20; DFP25].

Explicit formulas in the when X is a shot-noise process, i.e., $r(x) = cx$ [BM21b; BKM19].

Extensions to affine $r(x) = b + cx$ and two-sided jumps $\nu((-\infty, 0]) > 0$ [BM21a].

Our goals: Establish convergence rates to stationarity; tail bounds for stationary law for general release rate r non-negative ν .

Transience and null/positive recurrence

Regime	Conditions	Comments/Examples
Transient	(a) $m_{\mathbf{v}} < \infty$, $\limsup_{u \rightarrow \infty} \mathbf{r}(u) < m_{\mathbf{v}}$ (b) $m_{\mathbf{v}} = \infty$, $\liminf_{u \rightarrow \infty} \frac{u}{\mathbf{r}(u)} \int_0^\infty \frac{\bar{\mathbf{v}}(uv) dv}{1+v^2} > 1$	Ex: $\bar{\mathbf{v}}(u) \approx u^{-\alpha}$, $\mathbf{r}(u) \approx u^\beta$: transience if $\alpha + \beta < 1$
Null recurrent	$m_{\mathbf{v}} < \infty$, $\mathbf{r}(u) = m_{\mathbf{v}}$ for $u \geq u_0$	“mean zero” increments when X_t is large
Pos. recurrent	$\limsup_{u \rightarrow \infty} \int_0^\infty \frac{\bar{\mathbf{v}}(v)}{\mathbf{r}(u+v)} dv < 1$	Ex: $\alpha + \beta > 1$ with $\alpha, \beta \in (0, 1)$; and $m_{\mathbf{v}} < \liminf_{u \rightarrow \infty} \mathbf{r}(u)$

Here, $\mathbf{r}$ denotes the release rate, and $\bar{\mathbf{v}}(u) = \mathbf{v}((u, \infty))$, $m_{\mathbf{v}} = \int_{(0, \infty)} u \mathbf{v}(du)$ Lévy measure $\mathbf{v}$ of the input subordinator A .

Summary of main results: range of phenomena when $r(u) \approx u^\beta$

Summary of main results: range of phenomena when $r(u) \approx u^\beta$

Release rate $r(u)$ as $u \uparrow \infty$	Input Rate $\bar{v}(u)$ as $u \uparrow \infty$	Conv. Rate in TV as $t \uparrow \infty$	$\bar{\pi}(u)$ as $u \uparrow \infty$
$r(u) = a$ (large u) $m_v < a$	$\bar{v}(u) \preccurlyeq \exp(-cu)$	Exponential	Exponential
	$\bar{v}(u) \approx u^{-\alpha}, \alpha > 1$	$\approx t^{1-\alpha}$	$\approx u^{1-\alpha}$
$r(u) \approx u$	$\bar{v}(u) \approx u^{-\alpha}, \alpha > 0$	Exponential	$\approx u^{-\alpha}$
	$\bar{v}(u) \approx (\ln u)^{-\alpha}, \alpha > 1$	$\approx t^{1-\alpha}$	$\approx (\ln u)^{1-\alpha}$
$r(u) \approx u^\beta, \beta \in (0,1)$	$\bar{v}(u) \approx \exp(-u^{1-\beta})$	Exponential	$\approx \exp(-u^{1-\beta})$
	$\bar{v}(u) \approx u^{-\alpha}, \alpha + \beta > 1$	$\approx t^{\frac{1-\alpha-\beta}{1-\beta}}$	$\approx u^{1-\alpha-\beta}$
$r(u) \approx u^\beta, \beta \in (1,\infty)$	$\bar{v}(u) \preccurlyeq u^{-\alpha}, \alpha > 0$	Uniformly Ergodic	$\preccurlyeq u^{1-\alpha-\beta}$

$\preccurlyeq$: upper asymptotics; $\approx$: matching upper/lower bounds; $m_v := \int_{[0,\infty)} u v(du)$

π denotes the invariant measure for X with $\bar{\pi}(u) = \pi((u,\infty))$ the corresponding tail function.

The “Convergence Rate” column quantifies $\|\mathbb{P}^x(X_t \in \cdot) - \pi\|_{TV}$, as $t \rightarrow \infty$.

Theorem 2.2: Positive recurrent ergodicity rates (Part i: Upper bounds)

Assume that X is irreducible, aperiodic, and that all compact sets are [petite](#).

Theorem 2.2: Positive recurrent ergodicity rates (Part i: Upper bounds)

Assume that X is irreducible, aperiodic, and that all compact sets are **petite**.

Suppose there exist:

Lyapunov function $V : [0, \infty) \rightarrow [1, \infty)$.

Concave, non-decreasing function $\phi : [1, \infty) \rightarrow (0, \infty)$ satisfying $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, such that:

$$\limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1.$$

Theorem 2.2: Positive recurrent ergodicity rates (Part i: Upper bounds)

Assume that X is irreducible, aperiodic, and that all compact sets are petite.

Suppose there exist:

Lyapunov function $V : [0, \infty) \rightarrow [1, \infty)$.

Concave, non-decreasing function $\phi : [1, \infty) \rightarrow (0, \infty)$ satisfying $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, such that:

$$\limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1.$$

Then it follows:

$\int_{[0, \infty)} \phi \circ V(x) \pi(dx) < \infty$ and $\bar{\pi}(u) = \pi((u, \infty)) \leq C_\pi / \phi \circ V(u)$ for some $C_\pi \in (0, \infty)$.

Theorem 2.2: Positive recurrent ergodicity rates (Part i: Upper bounds)

Assume that X is irreducible, aperiodic, and that all compact sets are petite.

Suppose there exist:

Lyapunov function $V : [0, \infty) \rightarrow [1, \infty)$.

Concave, non-decreasing function $\phi : [1, \infty) \rightarrow (0, \infty)$ satisfying $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, such that:

$$\limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1.$$

Then it follows:

$\int_{[0, \infty)} \phi \circ V(x) \pi(dx) < \infty$ and $\bar{\pi}(u) = \pi((u, \infty)) \leq C_\pi / \phi \circ V(u)$ for some $C_\pi \in (0, \infty)$.

For any $x \in [0, \infty)$, there exists a constant $C_{TV} \in (0, \infty)$, such that:

$$\|p(t, x, dy) - \pi(dy)\|_{TV} \leq \frac{C_{TV}}{\phi(\Phi^{-1}(t))}, \quad \Phi(t) := \int_1^t \frac{ds}{\phi(s)}.$$

Positive recurrent ergodicity rates (Part ii: Lower bounds)

Assume X is ergodic with stationary measure π :

$u \mapsto r(u)/(u \log u)$ is eventually **decreasing** and $\lim_{u \rightarrow \infty} r(u)/(u \log u) = 0$.

$u \mapsto 1/\bar{v}(u)$ is **submultiplicative**.

Positive recurrent ergodicity rates (Part ii: Lower bounds)

Assume X is ergodic with stationary measure π :

$u \mapsto r(u)/(u \log u)$ is eventually **decreasing** and $\lim_{u \rightarrow \infty} r(u)/(u \log u) = 0$.

$u \mapsto 1/\bar{v}(u)$ is **submultiplicative**.

Then, we obtain:

$$\bar{\pi}(u) \geq L_\varepsilon(u), \quad \forall u \geq 1 \quad \text{where} \quad L_\varepsilon(u) := c_\varepsilon u^{1-\varepsilon} \frac{\bar{v}(u)}{r(u)}$$

Positive recurrent ergodicity rates (Part ii: Lower bounds)

Assume X is ergodic with stationary measure π :

$u \mapsto r(u)/(u \log u)$ is eventually **decreasing** and $\lim_{u \rightarrow \infty} r(u)/(u \log u) = 0$.

$u \mapsto 1/\bar{v}(u)$ is **submultiplicative**.

Then, we obtain:

$$\bar{\pi}(u) \geq L_\varepsilon(u), \quad \forall u \geq 1 \quad \text{where} \quad L_\varepsilon(u) := c_\varepsilon u^{1-\varepsilon} \frac{\bar{v}(u)}{r(u)}$$

Furthermore, assume there exists an increasing $h: [1, \infty) \rightarrow [1, \infty)$ and constant $c > 0$ such that

$$\mathbb{E}^x[h(X_t)] \leq h(x) + ct, \quad \forall x \geq 0, t \geq 1, \quad \text{and} \quad \lim_{u \rightarrow \infty} F(u) = \infty \quad \text{where} \quad F(u) := u L_\varepsilon(h^{-1}(u)).$$

Positive recurrent ergodicity rates (Part ii: Lower bounds)

Assume X is ergodic with stationary measure π :

$u \mapsto r(u)/(u \log u)$ is eventually **decreasing** and $\lim_{u \rightarrow \infty} r(u)/(u \log u) = 0$.

$u \mapsto 1/\bar{v}(u)$ is **submultiplicative**.

Then, we obtain:

$$\bar{\pi}(u) \geq L_\varepsilon(u), \quad \forall u \geq 1 \quad \text{where} \quad L_\varepsilon(u) := c_\varepsilon u^{1-\varepsilon} \frac{\bar{v}(u)}{r(u)}$$

Furthermore, assume there exists an increasing $h: [1, \infty) \rightarrow [1, \infty)$ and constant $c > 0$ such that

$$\mathbb{E}^x[h(X_t)] \leq h(x) + ct, \quad \forall x \geq 0, t \geq 1, \quad \text{and} \quad \lim_{u \rightarrow \infty} F(u) = \infty \quad \text{where} \quad F(u) := u L_\varepsilon(h^{-1}(u)).$$

The lower bound on the convergence rate follows:

$$\|p(t, x, dy) - \pi(dy)\|_{\text{TV}} \geq \frac{1}{2} L_\varepsilon \circ h^{-1} \circ F^{-1}(2(h(x) + ct)), \quad \forall x \geq 0.$$

In heavy-tail regimes, these **match** the upper bounds from part(i), giving **sharp** rates and tail behaviour.

Proof idea — Upper bounds

Goal: derive bounds on **stability** from a **Lyapunov** drift condition via [DFG09].

Proof idea — Upper bounds

Goal: derive bounds on **stability** from a **Lyapunov** drift condition via [DFG09].

$$\text{Assumption} \quad \limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1, \quad \text{yields:}$$

$$\mathcal{L}V(u) \leq -\phi \circ V(u) + c1_D(u),$$

where ϕ is concave and increasing, D compact (petite).

Proof idea — Upper bounds

Goal: derive bounds on **stability** from a **Lyapunov** drift condition via [DFG09].

$$\text{Assumption} \quad \limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1, \quad \text{yields:}$$

$$\mathcal{L}V(u) \leq -\phi \circ V(u) + c 1_D(u),$$

where ϕ is concave and increasing, D compact (petite).

The function $\phi \circ V$ is π -integrable: $\int_{[0,\infty)} \phi \circ V(x) \pi(dx) < \infty$.

Proof idea — Upper bounds

Goal: derive bounds on **stability** from a **Lyapunov** drift condition via [DFG09].

$$\text{Assumption} \quad \limsup_{u \rightarrow \infty} \frac{\int_0^\infty \frac{\phi(V(u+v))}{r(u+v)} \bar{v}(v) dv}{\phi(V(u))} < 1, \quad \text{yields:}$$

$$\mathcal{L}V(u) \leq -\phi \circ V(u) + c 1_D(u),$$

where ϕ is concave and increasing, D compact (petite).

The function $\phi \circ V$ is π -integrable: $\int_{[0,\infty)} \phi \circ V(x) \pi(dx) < \infty$.

The function $\Phi(t) = \int_1^t \frac{ds}{\phi(s)}$ controls return-time growth.

Using the regeneration structure, one obtains

$$\|p(t, x, dy) - \pi(dy)\|_{\text{TV}} \leq \frac{C_{\text{TV}}}{\phi(\Phi^{-1}(t))}$$

giving exponential or polynomial rates depending on ϕ .

Proof idea — Lower bounds

Let $V = u$ on the complement of the origin. Then $\mathcal{L}(1/V) \leq r(u)/u^2 =: \phi(1/u)$ for u large.

Proof idea — Lower bounds

Let $V = u$ on the complement of the origin. Then $\mathcal{L}(1/V) \leq r(u)/u^2 =: \varphi(1/u)$ for u large. $\varphi(1/u)$ is a proxy for the **duration** of the excursion from a compact set.

Proof idea — Lower bounds

Let $V = u$ on the complement of the origin. Then $\mathcal{L}(1/V) \leq r(u)/u^2 =: \varphi(1/u)$ for u large. $\varphi(1/u)$ is a proxy for the **duration** of the excursion from a compact set.

Define $\rho_r := \{t > 0 : V(X_t) \geq r\}$ and $\lambda_\ell := \{t > 0 : V(X_t) \leq \ell\}$. For, fixed $x \in (\ell, r)$, we have

$$\mathbb{P}_x(\rho_r < \lambda_\ell) \geq \bar{v}(r), \quad \text{as } r \rightarrow \infty, \quad \bar{v} \text{ bounds the size of the excursion.}$$

Proof idea — Lower bounds

Let $V = u$ on the complement of the origin. Then $\mathcal{L}(1/V) \leq r(u)/u^2 =: \varphi(1/u)$ for u large. $\varphi(1/u)$ is a proxy for the **duration** of the excursion from a compact set.

Define $\rho_r := \{t > 0 : V(X_t) \geq r\}$ and $\lambda_\ell := \{t > 0 : V(X_t) \leq \ell\}$. For, fixed $x \in (\ell, r)$, we have

$$\mathbb{P}_x(\rho_r < \lambda_\ell) \geq \bar{v}(r), \quad \text{as } r \rightarrow \infty, \quad \bar{v} \text{ bounds the size of the excursion.}$$

This implies that X satisfies the **L**-drift conditions $L(V, \varphi, \bar{v})$ in [BM24], which yields

$$\bar{\pi}(u) \geq c_\varepsilon u^{1-\varepsilon} \frac{\bar{v}(u)}{r(u)}, \quad u \geq 1$$

Proof idea — Lower bounds (2)

$$\bar{\pi}(u) \geq c_\varepsilon u^{1-\varepsilon} \frac{\bar{v}(u)}{r(u)}, \quad u \geq 1$$

There exists an increasing $h : [1, \infty) \rightarrow [1, \infty)$ and constant $c > 0$ such that

$$\mathbb{E}^x[h(X_t)] \leq h(x) + ct, \quad \forall x \geq 0, t \geq 1, \quad F(u) := u L_\varepsilon(h^{-1}(u)).$$

Lower bound on the convergence rate:

$$\|p(t, x, dy) - \pi(dy)\|_{\text{TV}} \geq \frac{1}{2} L_\varepsilon \circ h^{-1} \circ F^{-1}(2(h(x) + ct)), \quad \forall x \geq 0.$$

- [BKM19] O. Boxma, O. Kella, and M. Mandjes. "Infinite-server systems with Coxian arrivals". *Queueing Syst.* 92.3-4 (2019), pp. 233–255. ISSN: 0257-0130. DOI: 10.1007/s11134-019-09613-2. URL: <https://doi.org/10.1007/s11134-019-09613-2>.
- [BM21a] O. Boxma and M. Mandjes. "Affine storage and insurance risk models". *Math. Oper. Res.* 46.4 (2021), pp. 1282–1302. ISSN: 0364-765X. DOI: 10.1287/moor.2020.1097. URL: <https://doi.org/10.1287/moor.2020.1097>.
- [BM21b] O. Boxma and M. Mandjes. "Shot-noise queueing models". *Queueing Syst.* 99.1-2 (2021), pp. 121–159. ISSN: 0257-0130. DOI: 10.1007/s11134-021-09708-9. URL: <https://doi.org/10.1007/s11134-021-09708-9>.
- [BM24] M. Brešar and A. Mijatović. "Subexponential lower bounds for f -ergodic Markov processes". *To appear in Probab. Theory Related Fields* (2024), 58 pp.
- [BRT82] P. J. Brockwell, S. I. Resnick, and R. L. Tweedie. "Storage processes with general release rule and additive inputs". *Adv. in Appl. Probab.* 14.2 (1982), pp. 392–433.
- [DFG09] R. Douc, G. Fort, and A. Guillin. "Subgeometric rates of convergence of f -ergodic strong Markov processes". *Stochastic Process. Appl.* 119.3 (2009), pp. 897–923.
- [DFP25] A. Daw, B. Fralix, and J. Pender. "Establishing convergence of infinite-server queues with batch arrivals to shot-noise processes". *Oper. Res.* 73.4 (2025), pp. 2002–2009. ISSN: 0030-364X.
- [HR76] J. M. Harrison and S. I. Resnick. "The stationary distribution and first exit probabilities of a storage process with general release rule". *Math. Oper. Res.* 1.4 (1976), pp. 347–358.
- [MT93] S. P. Meyn and R. L. Tweedie. "Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes". *Adv. in Appl. Probab.* 25.3 (1993), pp. 518–548.
- [SBM20] M. Saxena, O. Boxma, and M. Mandjes. "An infinite-server system with Lévy shot-noise modulation: moments and asymptotics". *Markov Process. Related Fields* 26.4 (2020), pp. 757–778. ISSN: 1024-2953.
- [Yam05] M. Yamazato. "Recurrence-transience criteria for storage processes and their applications". *Tokyo J. Math.* 28.2 (2005), pp. 309–330.

Thank you!