

Critical branching processes with immigration: scaling limits of local extinction sets

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Part I: Results

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Bienaymé-Galton-Watson process with immigration (BGWI)

Offspring μ ($\sim (\xi_i^{(n)})_{i,n \in \mathbb{N}}$ IID) & **Immigration** ν ($\sim (\eta_n)_{n \geq 1}$ IID) laws on $\mathbb{N} := \{0, 1, \dots\}$, satisfying $(\xi_i^{(n)})_{i,n \in \mathbb{N}} \perp\!\!\!\perp (\eta_n)_{n \geq 1}$:

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Main Question. Is there a scaling limit of L_1 ? More generally, is there joint scaling limit

$$\left(\frac{1}{b_n} Z_1(\lfloor n \cdot \rfloor), \frac{1}{c_n} L_1(\lfloor n \cdot \rfloor) \right) \quad \text{for some scaling sequences } b_n, c_n \rightarrow \infty?$$

Discrete Lamperti transformation of BGWI & the scaling limit for Z_1

The formula $Z_1(0) = z \in \mathbb{N}$ & $Z_1(n+1) = \sum_{i=1}^{Z_1(n)} \xi_i^{(n)} + \eta_{n+1}$, $n \geq 1$, implies

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random walks $X_1 \perp Y_1$, have jump laws $\tilde{\mu}$ ($\tilde{\mu}(k) = \mu(k+1)$, $k \in \mathbb{N} \cup \{-1\}$) and ν . Note:

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Heuristic: if X_1 and Y_1 have scaling limits, Z_1 will be in a domain of attraction (DoA) of a self-similar continuous-state branching process with immigration (CBI).

Caution: scale X_1 and Y_1 in a **balanced** way so neither reproduction nor immigration dominate.

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Assumption (SL) The generating functions

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for $c, d > 0$ and $\alpha \in (0, 1]$ and slowly varying functions $\ell, k : (0, 1) \rightarrow (0, \infty)$, s.t. $\frac{\ell(s)}{k(s)} \xrightarrow{s \rightarrow 0} 1$

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Theorem ([CPGUB13]: $X_1 \in \text{RW}(\tilde{\mu})$, $Y_1 \in \text{RW}(\nu)$, $Z_1(n) = X_1(\sum_{i=1}^{n-1} Z_1(i)) + Y_1(n)$)

- ◇ $(X_1(\lfloor nb_n \cdot \rfloor), Y_1(\lfloor n \cdot \rfloor)) / b_n \xrightarrow{d} (X, Y)$, where $b_n^\alpha \sim n \ell(1/b_n) \sim n \bar{k}(1/b_n)$ as $n \rightarrow \infty$, α -stable subordinator Y and $(1+\alpha)$ -stable spectrally positive Lévy process X ;

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- $Z_1(\lfloor n \cdot \rfloor) / b_n \xrightarrow{d} Z$, where $Z = X(\int_0^\cdot Z(s) ds) + Y$ is a self-similar CBI of index α .

Joint scaling limit for BGWI Z_1 and counting local time L_1 under (SL)

Assumption (SL) The generating functions for reproduction μ and immigration ν :

$$f(s) = s + c(1-s)^{1+\alpha}\ell(1-s) \quad \& \quad g(s) = 1 - d(1-s)^\alpha k(1-s),$$

where $\ell(s) \sim k(s)$ as $s \rightarrow 0$ and $c, d > 0$, $0 < \alpha \leq 1$.

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Theorem (Main Limit Theorem in [MPUB25])

Let Assumption (SL) hold with $\delta = \frac{d}{\alpha c} \in (0, 1)$. Then, for any $\kappa > 0$, the sequence (c_n) ,

$$c_n := \kappa n \mathbb{P}(Z_1(n) = 0), \quad n \in \mathbb{N},$$

is regularly varying of index $1 - \delta$. There exists κ so that weak convergence (in Skorohod top.)

$$\left(\frac{1}{b_n} Z_1(\lfloor n \cdot \rfloor), \frac{1}{c_n} L_1(\lfloor n \cdot \rfloor) \right) \xrightarrow{d} (Z, L) \text{ as } n \rightarrow \infty$$

holds, where $b_n^\alpha \sim n\ell(1/b_n)$, $Z = X(\int_0^\cdot Z(s) ds) + Y$ α -self-similar CBI (α -stable subord. Y , $(1 + \alpha)$ -stable spectrally positive Lévy process X) and L Markov local time of Z at 0.

Remarks on the Main Limit Theorem

Under (SL) and $\delta = \frac{d}{\alpha c} \in (0, 1)$, $\exists \kappa > 0$ s.t. regularly varying $c_n := \kappa n \mathbb{P}(Z_1(n) = 0)$ satisfies

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- ◇ $\delta \in (0, 1)$ is necessary: if $\delta \geq 1$, Z is not point recurrent at 0 [FUB14]. For example, if $\alpha = 1$, then $Z \sim \frac{2}{c} \cdot \text{BESQ}(2\delta)$, making 0 is polar.

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- ◇ **Weak limit (1) covers all** critical offspring and immigration laws μ and ν in $\text{SDoA}(1 + \alpha)$ and $\text{SDoA}(\alpha)$, respectively, for $\alpha \in (0, 1)$.

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- ◇ **Weak limit (1) covers all** critical finite-variance offspring laws μ . Reproduction μ has finite variance $\iff f''(1) < \infty \iff \alpha = 1$ & $\exists \lim_{s \downarrow 0} \ell(s) < \infty$. Then, by balance condition $\frac{\ell(s)}{k(s)} \rightarrow 1$ as $s \rightarrow 0$ in Assumption (SL), we have finite-mean immigration ν .

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- If $\alpha = 1$ but $f''(1) = \infty$, offspring law μ has **infinite variance** but scaling limit (1) equals $\frac{2}{c} \cdot \text{BESQ}(2\delta)$ and its Markovian local time at 0.

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- If $\alpha = 1$ but $f''(1) = \infty$, offspring law μ has **infinite variance** but scaling limit (1) equals $\frac{2}{c} \cdot \text{BESQ}(2\delta)$ and its Markovian local time at 0.
- Substituting (μ, ν) with (μ, ν_n) , for immigration $\nu_n := (1 - p_n)\delta_0 + p_n \frac{1}{1 - \nu(0)} \nu|_{\mathbb{N} \setminus \{0\}}$ with $p_n \uparrow 1$, produces $\frac{1}{b_n} Z_1^{(n)}(\lfloor n \cdot \rfloor) \xrightarrow{d} Z$ as in (1), but $\frac{1}{c'_n} L_1^{(n)}(\lfloor n \cdot \rfloor) \not\xrightarrow{d} L$ for any scaling $(c'_n)_{n \geq 1}$.

Septuple Limit Theorem

- ◇ $Z_1 = X_1 \circ C_1 + Y_1$, where $C_1 = \sum_{0 \leq j < \cdot} Z_1(j)$ and $\text{RW}(\tilde{\mu}) \ni X_1 \perp\!\!\!\perp Y_1 \in \text{RW}(\nu)$.
- ◇ Let $L_1(X_1)$ and $L_1(Z_1)$ be the counting local times at 0 of X_1 and Z_1 .

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- ◇ Let $L_1(X_1)$ and $L_1(Z_1)$ be the counting local times at 0 of X_1 and Z_1 .
- ◇ $Z = X \circ C + Y$, where $C = \int_0^\cdot Z(s) ds$ and $X \perp\!\!\!\perp Y$ stable Lévy processes with Laplace exponents $\Psi(\lambda) = c\lambda^{1+\alpha}$ and $\Phi(\lambda) = d\lambda^\alpha$.
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Theorem (Septuple limit theorem in [MPUB25])

Under (SL) with $\delta = \frac{d}{\alpha c} \in (0, 1)$ and let $(b_n), (c_n)$ be regularly varying scaling sequences as above. Then there exists a regularly varying sequence (a_n) of index $1 + 1/\alpha$, such that

$$\left(\frac{X_1(\lfloor nb_n \cdot \rfloor)}{b_n}, \frac{L_1(X_1)(\lfloor nb_n \cdot \rfloor)}{a_n b_n}, \frac{Y_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{C_1(\lfloor n \cdot \rfloor)}{n b_n}, \frac{X_1 \circ C_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{Z_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{L_1(Z_1)(\lfloor n \cdot \rfloor)}{c_n} \right) \xrightarrow{d} (X, L(X), Y, C, X \circ C, Z, L(Z)) \quad \text{as } n \rightarrow \infty.$$

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The **Septuple Limit Theorem** follows directly from the **Main Limit Theorem** [MPUB25], **time-change equations** and their stability properties for CBIs [CPGUB13] and the **invariance principle for Local times** [MUB22] applied to the downward skip-free random walk X_1 .

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Proof of the **Main Limit Theorem** [MPUB25] is based on the **invariance principle for Local times** [MUB22] applied to the BGWI Z_1 :

- ◇ The proof of **Main Limit Theorem** presents a new **paradigm** for the verification of the assumptions of the invariance principle for local times [MUB22] that might have applications in other settings (reinforced RWs, RWs in random environment, etc).

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- ◇ The verification of the assumptions in [MUB22] for BGWI Z_1 is non-trivial, requiring novel results on branching processes and their scaling limits.
- ◇ **Part II** of this presentation describes this **paradigm**.

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Thank you for your attention!!
[MPUB25] available on arXiv.