

Critical branching processes with immigration: scaling limits of local extinction sets

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Part II: Description of the Proof

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Scaling limit for BGWI process Z_1 requires Assumption (SL)

Assumption (SL) The generating functions

$$f(s) := \sum_{n \in \mathbb{N}} s^n \mu(n) \quad \text{and} \quad g(s) := \sum_{n \in \mathbb{N}} s^n \nu(n), \quad s \in [0, 1],$$

of the offspring and immigration distributions μ and ν on $\mathbb{N} := \{0, 1, \dots\}$ take the form

$$f(s) = s + c(1-s)^{1+\alpha} \ell(1-s) \quad \text{and} \quad g(s) = 1 - d(1-s)^\alpha k(1-s), \quad s \in (0, 1),$$

for $c, d > 0$ and $\alpha \in (0, 1]$ and slowly varying functions $\ell, k : (0, 1) \rightarrow (0, \infty)$, s.t. $\frac{\ell(s)}{k(s)} \xrightarrow{s \rightarrow 0} 1$

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Theorem ([Li06]) $X_1 \in \text{RW}(\tilde{\mu})$, $Y_1 \in \text{RW}(\nu)$, $Z_1(n) = X_1(\sum_{i=1}^{n-1} Z_1(i)) + Y_1(n)$

- $(X_1(\lfloor nb_n \cdot \rfloor), Y_1(\lfloor n \cdot \rfloor)) / b_n \xrightarrow{d} (X, Y)$, where $b_n^\alpha \sim n \ell(1/b_n) \sim n \bar{k}(1/b_n)$ as $n \rightarrow \infty$, α -stable subordinator Y and $(1+\alpha)$ -stable spectrally positive Lévy process X ;
- $Z_1(\lfloor n \cdot \rfloor) / b_n \xrightarrow{d} Z$, where $Z = X(\int_0^\cdot Z(s) ds) + Y$ is a self-similar CBI of index α .

Joint scaling limit for BGWI Z_1 and counting local time L_1 under (SL)

Assumption (SL) The generating functions for reproduction μ and immigration ν :

$$f(s) = s + c(1-s)^{1+\alpha}\ell(1-s) \quad \& \quad g(s) = 1 - d(1-s)^\alpha k(1-s),$$

where $\ell(s) \sim k(s)$ as $s \rightarrow 0$ and $c, d > 0$, $0 < \alpha \leq 1$.

Theorem (Main Limit Theorem in [MPUB25])

Let Assumption (SL) hold with $\delta = \frac{d}{\alpha c} \in (0, 1)$. Then, for any $\kappa > 0$, the sequence (c_n) ,

$$c_n := \kappa n \mathbb{P}(Z_1(n) = 0), \quad n \in \mathbb{N},$$

is regularly varying of index $1 - \delta$. There exists κ so that weak convergence (in Skorohod top.)

$$\left(\frac{1}{b_n} Z_1(\lfloor n \cdot \rfloor), \frac{1}{c_n} L_1(\lfloor n \cdot \rfloor) \right) \xrightarrow{d} (Z, L) \text{ as } n \rightarrow \infty$$

holds, where $b_n^\alpha \sim n\ell(1/b_n)$, $Z = X(\int_0^\cdot Z(s) ds) + Y$ α -self-similar CBI (α -stable subord. Y , $(1 + \alpha)$ -stable spectrally positive Lévy process X) and L Markov local time of Z at 0.

Invariance principle for local times [MUB22] in the language of BGWIs

Theorem ([MUB22, Thm 1])

Assumption (SL) holds with $\delta = \frac{d}{\alpha c} \in (0, 1)$ and $b_n^\alpha \sim n^{\ell}(1/b_n)$ as $n \rightarrow \infty$. Let sequence of scaled BGWIs $(Z_n \stackrel{d}{=} Z_1(\lfloor n \cdot \rfloor)/b_n)$ be coupled with self-similar CBI Z . For any $t \in [0, \infty)$, let

$$g_t(Z) := \sup \{s \leq t : Z(s) = 0\} \quad \text{and} \quad d_t(Z) := \inf \{s > t : Z(s) = 0\},$$

and the corresponding $g_t(Z_n)$ and $d_t(Z_n)$. Assume $Z_n \xrightarrow{\mathbb{P}} Z$ and, for every $t > 0$,

$$g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z) \quad \text{and} \quad d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z), \text{ as } n \rightarrow \infty. \quad (1)$$

Then for $L_n(t) := |\{s \in \mathbb{N}/n : Z_n(s) = 0\} \cap [0, t]|$, some $\tilde{\kappa} > 0$ and $\tilde{c}_n := \frac{\tilde{\kappa}}{\mathbb{P}(d_{1/n}(Z_n) > 1)}$, the following limit in probability holds:

$$(Z_n, L_n/\tilde{c}_n) \xrightarrow{\mathbb{P}} (Z, L) \text{ as } n \rightarrow \infty.$$

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Task: for a coupling (Z_n, Z) , prove (1) and show $\tilde{c}_n \sim c_n = \kappa n \mathbb{P}(Z_1(n) = 0)$ as $n \rightarrow \infty$.

Proving $g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z)$ and $d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z)$ for examples in [MUB22]

The applications in [MUB22] of the general form of Theorem on the previous slide for regenerative processes use path-wise arguments on the Skorokhod space to prove

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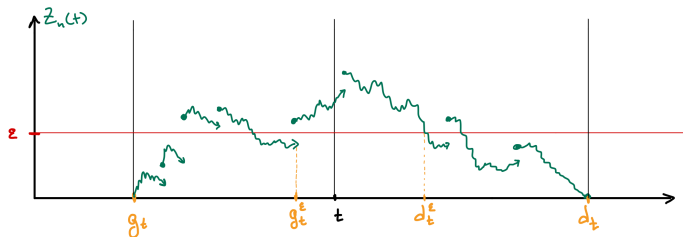
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A different approach is required for BGWIs!!

Proving $g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z)$ and $d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z)$ for branching processes

For $\varepsilon > 0$, for excursions of Z and Z_n straddling time $t > 0$, set

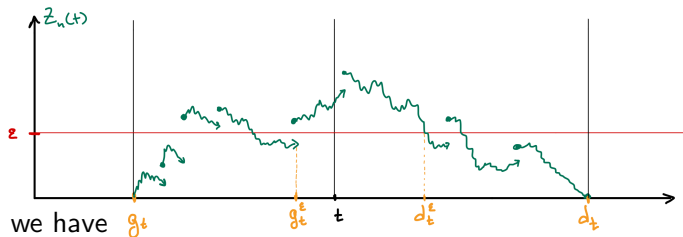
$$g_t^\varepsilon(Z) := \sup \{s \leq t : Z(s) \in [0, \varepsilon]\} \quad \text{and} \quad d_t^\varepsilon(Z) := \inf \{s > t : Z(s) \in [0, \varepsilon]\}.$$



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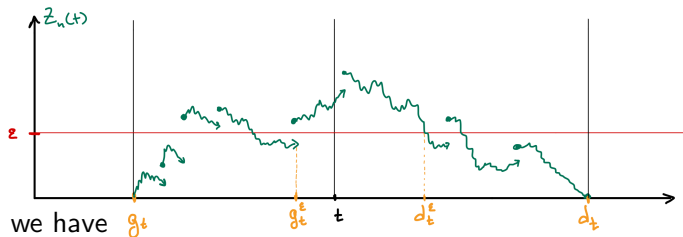
Then, for any $\eta > 0$, we have

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Then, for any $\eta > 0$, we have

$$\mathbb{P}(|d_t(Z) - d_t(Z_n)| > \eta) \leq \text{I}_d + \text{II}_d + \text{III}_d \quad \text{and} \quad \mathbb{P}(|g_t(Z) - g_t(Z_n)| > \eta) \leq \text{I}_g + \text{II}_g + \text{III}_g,$$

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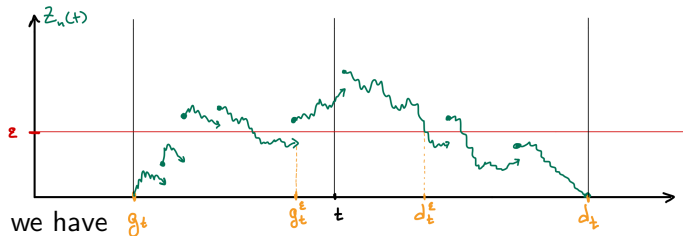
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We need to prove

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P(n, \varepsilon) = 0, \quad \text{where} \quad P(n, \varepsilon) \in \{\text{I}_d, \text{II}_d, \text{III}_d, \text{I}_g, \text{II}_g, \text{III}_g\}.$$

Proofs of I_g, I_d, II_g, II_d use path-wise arguments only

$$I_d = \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) \xrightarrow{\varepsilon \downarrow 0} 0 \quad \& \quad I_g = \mathbb{P}(|g_t(Z) - g_t^\varepsilon(Z)| > \eta/3) \xrightarrow{\varepsilon \downarrow 0} 0$$

require **quasi left continuity** of CBI Z and that $Z(s-) > 0$ for any $g_t(Z) < s < d_t(Z)$.

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Lemma

If $f \in D$ satisfies $T_O(f) \leq \min\{T_C(f), T_C^-(f)\}$, where $O = [0, \varepsilon)$, $C = [0, \varepsilon]$ and $T_O = \inf\{t \geq 0 : f(t) \in O\}$, $T_C^- = \inf\{t \geq 0 : f(t-) \in C\}$, then for any $f_n \rightarrow f$ we have $T_O(f_n) \rightarrow T_O(f)$.

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II_d : Z downwards regular at ε & Lemma $\implies |d_t^\varepsilon(Z) - d_t^\varepsilon(Z_n)| \xrightarrow{\text{a.s.}} 0$ as $n \uparrow \infty$

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II_g : time reverse $\hat{Z}(s) := Z(\max\{t-s, 0\}-)$, note that $g_t^\varepsilon(Z_n) = t - T_O(\hat{Z}_n)$ and $g_t^\varepsilon(Z) = t - T_O(\hat{Z})$ and apply Lemma to $\hat{Z}_n \rightarrow \hat{Z}$ to get $|g_t^\varepsilon(Z) - g_t^\varepsilon(Z_n)| \xrightarrow{\text{a.s.}} 0$ as $n \uparrow \infty$

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Theorem (Theorem 6 in [MPUB25])

Under Assumption (SL) with $\delta = \frac{d}{\alpha c} \in (0, 1)$, for any $t \geq 0$ and $\varepsilon > 0$, we have

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By [MPUB25, Thm 6] we get

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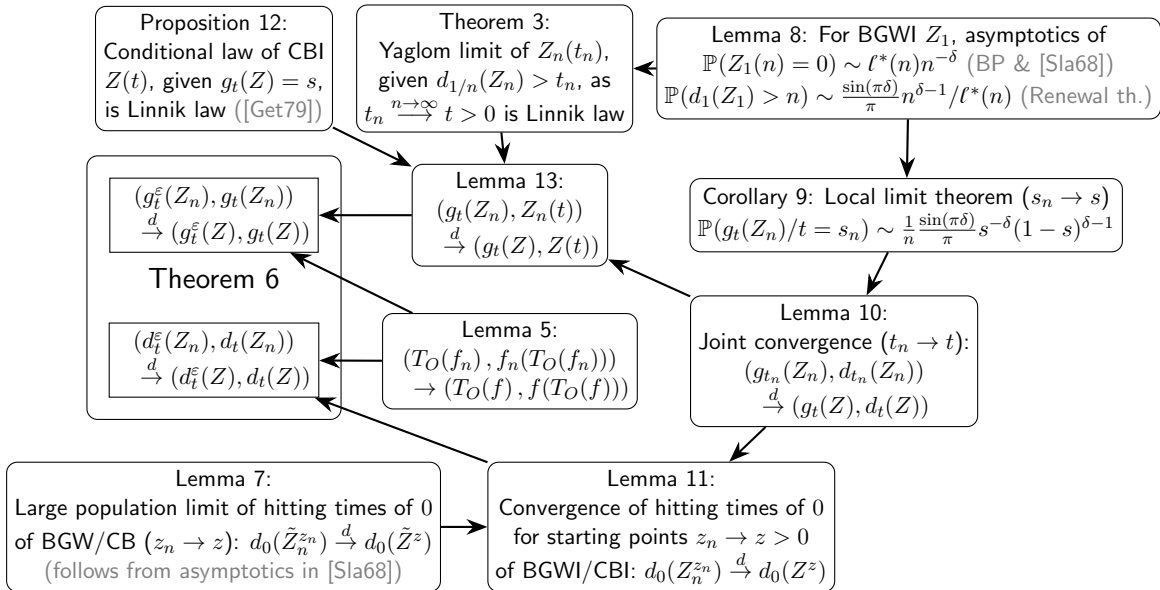
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By [MPUB25, Thm 6] we get

$$g_t^\varepsilon(Z_n) - g_t(Z_n) \xrightarrow{d} g_t^\varepsilon(Z) - g_t(Z) \quad \text{and} \quad d_t(Z_n) - d_t^\varepsilon(Z_n) \xrightarrow{d} d_t(Z) - d_t^\varepsilon(Z) \quad \text{as } n \rightarrow \infty$$

and hence $\text{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3) \xrightarrow{n \uparrow \infty} \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) = \text{I}_d$. Ditto for $\text{III}_g \xrightarrow{n \uparrow \infty} \text{I}_g$. We already know $\text{I}_d, \text{I}_g \xrightarrow{\varepsilon \downarrow \infty} 0$ finishing the proof.

But how does one prove [MPUB25, Thm 6]?



(numbers next to lemmas, corollary, proposition and theorems correspond to those in the preprint [MPUB25])

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Thank you for your attention!!
[MPUB25] available on arXiv.