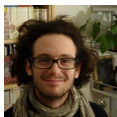


Central limit theorem for superdiffusive reflecting Brownian motion

Aleksandar Mijatović
Department of Statistics



Isao Sauzedde
University of Warwick



Andrew Wade
Durham University

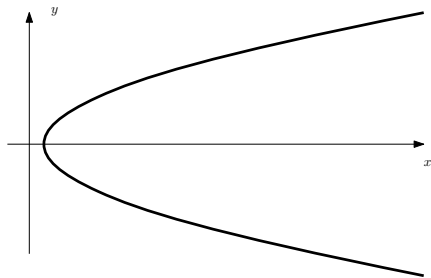
EPSRC

Engineering and Physical Sciences
Research Council

December 27, 2024

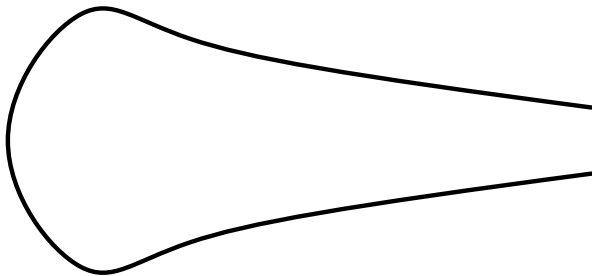
EP/W006227/1 & EP/V009478/1

The model: $\beta > 0$



$$\mathcal{D} := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}^d : |y|_d < b(x)\} \subset \mathbb{R}^{1+d}, \text{ where}$$
$$b(x) \sim_{x \rightarrow \infty} a_\infty x^\beta, \quad \text{for some } a_\infty > 0.$$

The model: $\beta < 0$



$$\mathcal{D} := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}^d : |y|_d < b(x)\} \subset \mathbb{R}^{1+d}, \text{ where}$$
$$b(x) \sim_{x \rightarrow \infty} a_\infty x^\beta, \quad \text{for some } a_\infty > 0.$$

The model

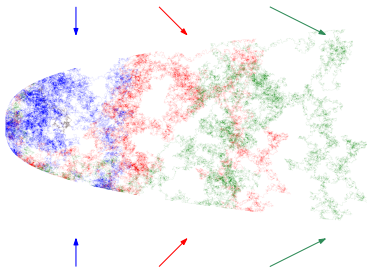
$$dZ_t = \sigma(Z_t) dW_t + \phi(Z_t) dL_t \quad \& \quad dL_t = \mathbb{1}_{Z_t \in \partial\mathcal{D}} dL_t \quad (1)$$

- ◇ W is a Brownian motion in $\mathbb{R}^{1+d}$
- ◇ L is local time of Z at $\partial\mathcal{D}$: non-decreasing cts process, increases only when $Z_t \in \partial\mathcal{D}$.
- ◇ [MMW23]: existence & strong uniqueness of SDER (1) (if σ uniformly elliptic & bounded).

The model

$$dZ_t = \sigma(Z_t) dW_t + \phi(Z_t) dL_t \quad \& \quad dL_t = \mathbb{1}_{Z_t \in \partial\mathcal{D}} dL_t \quad (1)$$

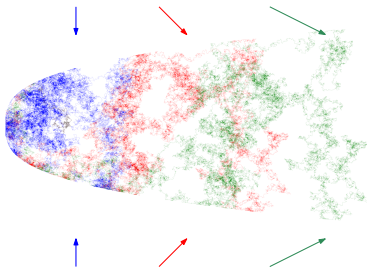
- ◇ W is a Brownian motion in $\mathbb{R}^{1+d}$
- ◇ L is local time of Z at $\partial\mathcal{D}$: non-decreasing cts process, increases only when $Z_t \in \partial\mathcal{D}$.
- ◇ [MMW23]: existence & strong uniqueness of SDER (1) (if σ uniformly elliptic & bounded).
- ◇ **No interior drift**, but **oblique reflection vector field** ϕ favours positive x (RBM, $\sigma = \text{Id}$):



The model

$$dZ_t = \sigma(Z_t) dW_t + \phi(Z_t) dL_t \quad \& \quad dL_t = \mathbb{1}_{Z_t \in \partial\mathcal{D}} dL_t \quad (1)$$

- ◇ W is a Brownian motion in $\mathbb{R}^{1+d}$
- ◇ L is local time of Z at $\partial\mathcal{D}$: non-decreasing cts process, increases only when $Z_t \in \partial\mathcal{D}$.
- ◇ [MMW23]: existence & strong uniqueness of SDER (1) (if σ uniformly elliptic & bounded).
- ◇ **No interior drift**, but **oblique reflection vector field** ϕ favours positive x (RBM, $\sigma = \text{Id}$):



Normal reflection vector field ϕ leads to a diffusive processes Z [BMW23].

Main result

[MMW23]: ϕ asymptotically oblique, then for $\beta \in (-1, 1)$, superdiffusive law of large numbers for the x -component of $Z = (X, Y)$ holds (constant $c_1 > 0$ explicit in model parameters):

$$X_T \sim c_1 T^{\frac{1}{1+\beta}} \quad \text{as } T \rightarrow \infty.$$

Main result

[MMW23]: ϕ asymptotically oblique, then for $\beta \in (-1, 1)$, superdiffusive law of large numbers for the x -component of $Z = (X, Y)$ holds (constant $c_1 > 0$ explicit in model parameters):

$$X_T \sim c_1 T^{\frac{1}{1+\beta}} \quad \text{as } T \rightarrow \infty.$$

Theorem ([MSW24] - Main Result)

Under mild assumptions, for $\beta \in (-1/3, 1)$, as $T \rightarrow \infty$,

$$\left(\frac{X_T - c_1 T^{\frac{1}{1+\beta}}}{\sqrt{T}}, \frac{Y_T}{a_\infty c_1^\beta T^{\beta/(1+\beta)}} \right) \longrightarrow \mathcal{N} \otimes \mu,$$

where $\mathcal{N}$ centred Gaussian on $\mathbb{R}$ with explicit variance Υ and μ a measure on ball $\mathbb{B}^d$ in $\mathbb{R}^d$.

Main result

[MMW23]: ϕ asymptotically oblique, then for $\beta \in (-1, 1)$, superdiffusive law of large numbers for the x -component of $Z = (X, Y)$ holds (constant $c_1 > 0$ explicit in model parameters):

$$X_T \sim c_1 T^{\frac{1}{1+\beta}} \quad \text{as } T \rightarrow \infty.$$

Theorem ([MSW24] - Main Result)

Under mild assumptions, for $\beta \in (-1/3, 1)$, as $T \rightarrow \infty$,

$$\left(\frac{X_T - c_1 T^{\frac{1}{1+\beta}}}{\sqrt{T}}, \frac{Y_T}{a_\infty c_1^\beta T^{\beta/(1+\beta)}} \right) \longrightarrow \mathcal{N} \otimes \mu,$$

where $\mathcal{N}$ centred Gaussian on $\mathbb{R}$ with explicit variance Υ and μ a measure on ball $\mathbb{B}^d$ in $\mathbb{R}^d$.

Recall $b(x) \sim a_\infty x^\beta$ as $x \rightarrow \infty$. Hence $b(X_T) \sim a_\infty c_1^\beta T^{\beta/(1+\beta)}$ a.s. as $T \rightarrow \infty$.

Main result

[MMW23]: ϕ asymptotically oblique, then for $\beta \in (-1, 1)$, superdiffusive law of large numbers for the x -component of $Z = (X, Y)$ holds (constant $c_1 > 0$ explicit in model parameters):

$$X_T \sim c_1 T^{\frac{1}{1+\beta}} \quad \text{as } T \rightarrow \infty.$$

Theorem ([MSW24] - Main Result)

Under mild assumptions, for $\beta \in (-1/3, 1)$, as $T \rightarrow \infty$,

$$\left(\frac{X_T - c_1 T^{\frac{1}{1+\beta}}}{\sqrt{T}}, \frac{Y_T}{a_\infty c_1^\beta T^{\beta/(1+\beta)}} \right) \longrightarrow \mathcal{N} \otimes \mu,$$

where $\mathcal{N}$ centred Gaussian on $\mathbb{R}$ with explicit variance Υ and μ a measure on ball $\mathbb{B}^d$ in $\mathbb{R}^d$.

Recall $b(x) \sim a_\infty x^\beta$ as $x \rightarrow \infty$. Hence $b(X_T) \sim a_\infty c_1^\beta T^{\beta/(1+\beta)}$ a.s. as $T \rightarrow \infty$.

For RBM ($\sigma = \text{Id}$), μ is uniform on $\mathbb{B}^d$ and variance of $\mathcal{N}$ equals $\Upsilon = 1 + \frac{s_0^2}{c_0^2} \frac{d}{d+2}$.

Weak limit of $Y_T/b(X_T)$ as $T \rightarrow \infty$ to invariant measure μ on $\mathbb{B}^d$

$$\mathcal{Z}^T := (\mathcal{Z}_t^T)_{t \geq 0}, \quad \text{where} \quad \mathcal{Z}_t^T := (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T) \quad (2)$$

Weak limit of $Y_T/b(X_T)$ as $T \rightarrow \infty$ to invariant measure μ on $\mathbb{B}^d$

$$\mathcal{Z}^T := (\mathcal{Z}_t^T)_{t \geq 0}, \quad \text{where} \quad \mathcal{Z}_t^T := (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T) \quad (2)$$

Theorem ([MSW24] - Local Process Convergence)

For $\beta \in (-1, 1)$, the process $\mathcal{Z}^T$ converges weakly (in locally uniform topology), as $T \rightarrow \infty$, to $\mathcal{Z}^\infty = (\mathcal{X}^\infty, \mathcal{Y}^\infty)$ in $\mathbb{R} \times \mathbb{B}^d$, where $\mathcal{X}^\infty = \pi_0 \sigma^\infty W + s_0 L^{\mathcal{Y}^\infty}$ and $\mathcal{Y}^\infty$ satisfies SDER on $\mathbb{B}^d$:

$$d\mathcal{Y}_t^\infty = \pi_d \sigma^\infty dW_t + \phi_\infty^{(d)}(\mathcal{Y}_t^\infty) dL_t^{\mathcal{Y}^\infty}, \quad (3)$$

where $\mathcal{X}_0^\infty = 0$ and $\mathcal{Y}_0^\infty$, following the invariant law μ of (3) on $\mathbb{B}^d$, is independent of W .

Weak limit of $Y_T/b(X_T)$ as $T \rightarrow \infty$ to invariant measure μ on $\mathbb{B}^d$

$$\mathcal{Z}^T := (\mathcal{Z}_t^T)_{t \geq 0}, \quad \text{where} \quad \mathcal{Z}_t^T := (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T) \quad (2)$$

Theorem ([MSW24] - Local Process Convergence)

For $\beta \in (-1, 1)$, the process $\mathcal{Z}^T$ converges weakly (in locally uniform topology), as $T \rightarrow \infty$, to $\mathcal{Z}^\infty = (\mathcal{X}^\infty, \mathcal{Y}^\infty)$ in $\mathbb{R} \times \mathbb{B}^d$, where $\mathcal{X}^\infty = \pi_0 \sigma^\infty W + s_0 L^{\mathcal{Y}^\infty}$ and $\mathcal{Y}^\infty$ satisfies SDER on $\mathbb{B}^d$:

$$d\mathcal{Y}_t^\infty = \pi_d \sigma^\infty dW_t + \phi_\infty^{(d)}(\mathcal{Y}_t^\infty) dL_t^{\mathcal{Y}^\infty}, \quad (3)$$

where $\mathcal{X}_0^\infty = 0$ and $\mathcal{Y}_0^\infty$, following the invariant law μ of (3) on $\mathbb{B}^d$, is independent of W .

$\exists s_0, c_0 > 0$ and vector field $\phi_\infty^{(d)} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ such that $\forall u \in \mathbb{S}^{d-1}$:

$$x^{\frac{1-\beta}{2}} \left| \phi(x, b(x)u) - (s_0, \phi_\infty^{(d)}(u)) \right|_{d+1} \xrightarrow{x \rightarrow \infty} 0 \quad \& \quad c_0 = \langle \phi_\infty^{(d)}(u), -u \rangle > 0.$$

Weak limit of $Y_T/b(X_T)$ as $T \rightarrow \infty$ to invariant measure μ on $\mathbb{B}^d$

$$\mathcal{Z}^T := (\mathcal{Z}_t^T)_{t \geq 0}, \quad \text{where} \quad \mathcal{Z}_t^T := (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T) \quad (2)$$

Theorem ([MSW24] - Local Process Convergence)

For $\beta \in (-1, 1)$, the process $\mathcal{Z}^T$ converges weakly (in locally uniform topology), as $T \rightarrow \infty$, to $\mathcal{Z}^\infty = (\mathcal{X}^\infty, \mathcal{Y}^\infty)$ in $\mathbb{R} \times \mathbb{B}^d$, where $\mathcal{X}^\infty = \pi_0 \sigma^\infty W + s_0 L^{\mathcal{Y}^\infty}$ and $\mathcal{Y}^\infty$ satisfies SDER on $\mathbb{B}^d$:

$$d\mathcal{Y}_t^\infty = \pi_d \sigma^\infty dW_t + \phi_\infty^{(d)}(\mathcal{Y}_t^\infty) dL_t^{\mathcal{Y}^\infty}, \quad (3)$$

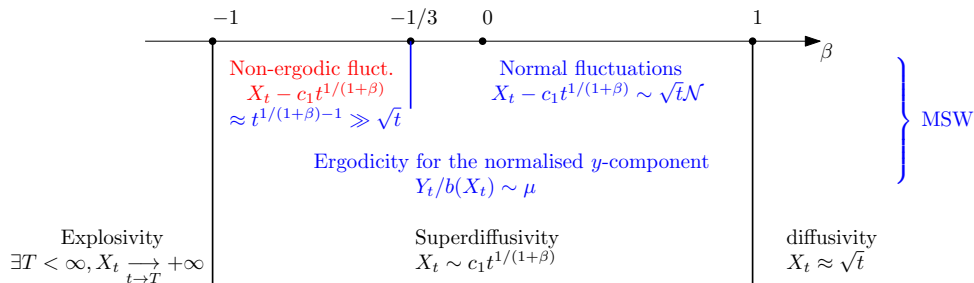
where $\mathcal{X}_0^\infty = 0$ and $\mathcal{Y}_0^\infty$, following the invariant law μ of (3) on $\mathbb{B}^d$, is independent of W .

$\exists s_0, c_0 > 0$ and vector field $\phi_\infty^{(d)} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ such that $\forall u \in \mathbb{S}^{d-1}$:

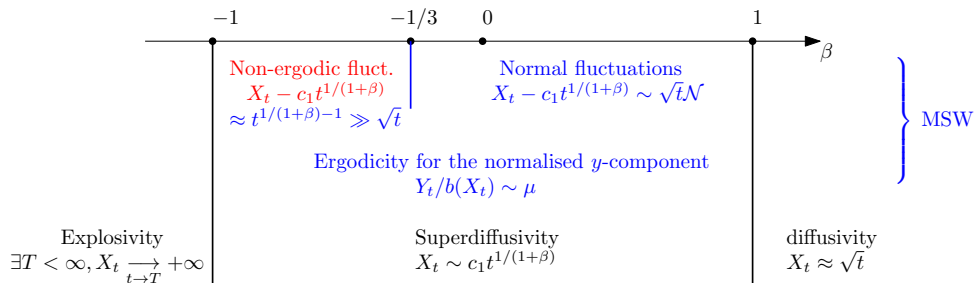
$$x^{\frac{1-\beta}{2}} \left| \phi(x, b(x)u) - (s_0, \phi_\infty^{(d)}(u)) \right|_{d+1} \xrightarrow{x \rightarrow \infty} 0 \quad \& \quad c_0 = \langle \phi_\infty^{(d)}(u), -u \rangle > 0.$$

σ^∞ is square root of $\Sigma^\infty := \lim_{x \rightarrow \infty} \sigma(z)\sigma(z)^\top$ (π_0 & π_d project $\mathbb{R}^{1+d}$ onto $\mathbb{R}$ and $\mathbb{R}^d$).

Phase diagram

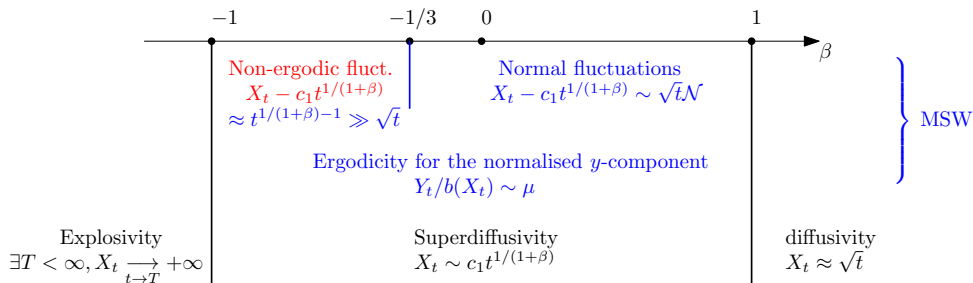


Phase diagram



Loss of CLT at $\beta = -1/3$ appears universal for X with $dX_t \approx X_t^{-\beta} dt + (L^2\text{-noise term})$.

Phase diagram



Loss of CLT at $\beta = -1/3$ appears universal for X with $dX_t \approx X_t^{-\beta} dt + (L^2\text{-noise term})$.
 If noise term α -stable, loss of CLT expected at $\beta = -1/(1 + \alpha)$

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

- ◇ Zoom around $x \approx c_1 T^{1/(1+\beta)}$, $T \gg 1$: domain $\approx$ infinite cylinder, coefficients $\approx$ constant, $\mathcal{Z}^\infty$ follows SDER (3) started at $\mathcal{Z}_0^\infty = (0, Y_T/b(X_T)) = \mathcal{Z}_0^T$, where

$$\mathcal{Z}_t^T = (X_{T+b(X_T)^2t} - X_T, Y_{T+b(X_T)^2t})/b(X_T), \quad t \geq 0.$$

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

- ◇ Zoom around $x \approx c_1 T^{1/(1+\beta)}$, $T \gg 1$: domain $\approx$ infinite cylinder, coefficients $\approx$ constant, $\mathcal{Z}^\infty$ follows SDER (3) started at $\mathcal{Z}_0^\infty = (0, Y_T/b(X_T)) = \mathcal{Z}_0^T$, where

$$\mathcal{Z}_t^T = (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T), \quad t \geq 0.$$

- ◇ Continuity result: $\mathcal{Z}^\infty \approx \mathcal{Z}^T$ if started from the same point. Stack domains of $\mathcal{Z}^T$ and $\mathbb{R} \times \mathbb{B}^d$ to get $(d+2)$ -dim domain and apply continuity in starting point [DI93]

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

- ◇ Zoom around $x \approx c_1 T^{1/(1+\beta)}$, $T \gg 1$: domain $\approx$ infinite cylinder, coefficients $\approx$ constant, $\mathcal{Z}^\infty$ follows SDER (3) started at $\mathcal{Z}_0^\infty = (0, Y_T/b(X_T)) = \mathcal{Z}_0^T$, where

$$\mathcal{Z}_t^T = (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T), \quad t \geq 0.$$

- ◇ Continuity result: $\mathcal{Z}^\infty \approx \mathcal{Z}^T$ if started from the same point. Stack domains of $\mathcal{Z}^T$ and $\mathbb{R} \times \mathbb{B}^d$ to get $(d+2)$ -dim domain and apply continuity in starting point [DI93]
- ◇ t large: $\mathcal{Y}_t^T \approx \mathcal{Y}_t^\infty \approx \mu$.

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

- Zoom around $x \approx c_1 T^{1/(1+\beta)}$, $T \gg 1$: domain $\approx$ infinite cylinder, coefficients $\approx$ constant, $\mathcal{Z}^\infty$ follows SDER (3) started at $\mathcal{Z}_0^\infty = (0, Y_T/b(X_T)) = \mathcal{Z}_0^T$, where

$$\mathcal{Z}_t^T = (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T), \quad t \geq 0.$$

- Continuity result: $\mathcal{Z}^\infty \approx \mathcal{Z}^T$ if started from the same point. Stack domains of $\mathcal{Z}^T$ and $\mathbb{R} \times \mathbb{B}^d$ to get $(d+2)$ -dim domain and apply continuity in starting point [DI93]
- t large: $\mathcal{Y}_t^T \approx \mathcal{Y}_t^\infty \approx \mu$.
- However for t too large: $\mathcal{Y}_t^T \not\approx \mathcal{Y}_t^\infty$. Approximation works up to $t \leq s(T)$, s not explicit.

Elements of the proof 1: convergence for the scaled y -component

First we proof the convergence for the y -component. Recall $X_T \sim c_1 T^{1/(1+\beta)}$.

- ◇ Zoom around $x \approx c_1 T^{1/(1+\beta)}$, $T \gg 1$: domain $\approx$ infinite cylinder, coefficients $\approx$ constant, $\mathcal{Z}^\infty$ follows SDER (3) started at $\mathcal{Z}_0^\infty = (0, Y_T/b(X_T)) = \mathcal{Z}_0^T$, where

$$\mathcal{Z}_t^T = (X_{T+b(X_T)^2 t} - X_T, Y_{T+b(X_T)^2 t})/b(X_T), \quad t \geq 0.$$

- ◇ Continuity result: $\mathcal{Z}^\infty \approx \mathcal{Z}^T$ if started from the same point. Stack domains of $\mathcal{Z}^T$ and $\mathbb{R} \times \mathbb{B}^d$ to get $(d+2)$ -dim domain and apply continuity in starting point [DI93]
- ◇ t large: $\mathcal{Y}_t^T \approx \mathcal{Y}_t^\infty \approx \mu$.
- ◇ However for t too large: $\mathcal{Y}_t^T \not\approx \mathcal{Y}_t^\infty$. Approximation works up to $t \leq s(T)$, s not explicit.
- ◇ A careful coupling construction over nested intervals of the processes $\mathcal{Z}^T$ and $\mathcal{Z}^\infty$ started at $(0, Y_T/b(X_T))$ and in stationarity is needed to conclude $Y_T/b(X_T) \rightarrow \mu$ weakly.

Elements of the proof 2: CLT for the x -component & asymptotic independence

Elements of the proof 2: CLT for the x -component & asymptotic independence

CLT

- ◇ Local time hard to control: use Lyapunov function f in [MMW23] to study $\xi_t = f(Z_t)$.

Elements of the proof 2: CLT for the x -component & asymptotic independence

CLT

- ◇ Local time hard to control: use Lyapunov function f in [MMW23] to study $\xi_t = f(Z_t)$.
- ◇ Finite variation part of ξ decomposes as (smooth drift term)+(small local time part), and such that $\xi_t = X_t^{1+\beta} + \text{small correction}$.

Elements of the proof 2: CLT for the x -component & asymptotic independence

CLT

- ◇ Local time hard to control: use Lyapunov function f in [MMW23] to study $\xi_t = f(Z_t)$.
- ◇ Finite variation part of ξ decomposes as (smooth drift term)+(small local time part), and such that $\xi_t = X_t^{1+\beta} + \text{small correction}$.
- ◇ Standard stochastic analysis used to control additive functionals $\int_0^t g(Z_s) \, ds$, where g is sufficiently nice. Using the convergence of $Y_T/b(X_T)$ and the law of large number for X , a type of ergodicity argument with rescaling yields a CLT from the martingale part of ξ .

Elements of the proof 2: CLT for the x -component & asymptotic independence

CLT

- ◇ Local time hard to control: use Lyapunov function f in [MMW23] to study $\xi_t = f(Z_t)$.
- ◇ Finite variation part of ξ decomposes as (smooth drift term)+(small local time part), and such that $\xi_t = X_t^{1+\beta} + \text{small correction}$.
- ◇ Standard stochastic analysis used to control additive functionals $\int_0^t g(Z_s) \, ds$, where g is sufficiently nice. Using the convergence of $Y_T/b(X_T)$ and the law of large number for X , a type of ergodicity argument with rescaling yields a CLT from the martingale part of ξ .

Asymptotic independence of $(X_T - c_1 T^{1/(\beta+1)})/\sqrt{T}$ and $Y_T/b(X_T)$:

Separation of time scales: $Y/b(X)$ mixes in a window of length $\approx T^{2\beta/(1+\beta)}$, during which $(X_t - c_1 t^{1/(1+\beta)})/\sqrt{t}$ does not move asymptotically as the window is pushed to infinity.

References

-  M. Brešar, A. Mijatović, and A. Wade, *Brownian motion with asymptotically normal reflection in unbounded domains: from transience to stability*, to appear in Ann. Probab. (2023), ArXiv: 2303.06916.
-  Paul Dupuis and Hitoshi Ishii, *SDEs with oblique reflection on nonsmooth domains*, Ann. Probab. **21** (1993), no. 1, 554–580. MR 1207237
-  Mikhail V. Menshikov, Aleksandar Mijatović, and Andrew R. Wade, *Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity*, Ann. Inst. Henri Poincaré Probab. Stat. **59** (2023), no. 4, 1813–1843. MR 4663508
-  A. Mijatović, I. Sauzedde, and A. Wade, *Central limit theorem for superdiffusive reflected Brownian motion*, ArXiv (2024), see link below the video.

Thank you for your attention!!
[MSW24] available on arXiv.