

Asymptotically optimal Wasserstein couplings for the small-time stable domain of attraction



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Classical setting: CLT and its rate of convergence

Let $(X_i)_{i \in \mathbb{N}}$ iid with $\mathbb{E}X_1^2 = 1$, $\mathbb{E}X_1 = 0$, then $S_n = \sum_{i=1}^n X_i$ satisfies

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$$\sup_{x \in \mathbb{R}} |\mathbb{P}(S_n/\sqrt{n} \leq x) - \mathbb{P}(M \leq x)| \leq C\mathbb{E}[|X_1|^3]/\sqrt{n} \quad \& \quad \mathcal{W}_1(S_n/\sqrt{n}, M) = \mathcal{O}(1/\sqrt{n}).$$

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What about **stable domain of attraction** in the **multi-dimensional small-time setting**?

Plan for this talk

- 1 Setting
- 2 Bounds & Rates
- 3 Couplings

Setting: small-time domain of attraction

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- $\mathbf{Z} = (\mathbf{Z}_t)_{t \in [0,1]}$ α -stable process in $\mathbb{R}^d$, i.e. self-similar Lévy process with $\mathbf{Z}_{mt} \sim m^{1/\alpha} \mathbf{Z}_t$ for all $m > 0$ and some stability index $\alpha \in (0, 2]$

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$$\mathbf{X}^t := (\mathbf{X}_{st}/g(t))_{s \in [0,1]} \xrightarrow{d} \mathbf{Z} \quad \text{as } t \downarrow 0$$

for $g(t) = t^{1/\alpha} G(1/t)$, where $G \in SV_\infty$ is *slowly varying* ($\forall c > 0: \lim_{x \rightarrow \infty} \frac{G(xc)}{G(x)} = 1$)

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Main question: what is the **rate** of convergence in Wasserstein distance of $\mathbf{X}^t \xrightarrow{d} \mathbf{Z}$ as $t \downarrow 0$?

q -Wasserstein distance $\mathcal{W}_q$ for any $q \in (0, \infty)$

For any $\mathbb{R}^d$ -valued stochastic processes $\mathbf{x} = (\mathbf{x}_t)_{t \in [0,1]}$ and $\mathbf{y} = (\mathbf{y}_t)_{t \in [0,1]}$,

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where infimum is over all couplings $(\mathbf{x}', \mathbf{y}')$ with $\mathbf{x}' \stackrel{d}{=} \mathbf{x}$ and $\mathbf{y}' \stackrel{d}{=} \mathbf{y}$ ($|\cdot|$ L^2 -norm on $\mathbb{R}^d$)

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where the infimum is taken over all couplings (ξ', ζ') with $\xi' \stackrel{d}{=} \xi$ and $\zeta' \stackrel{d}{=} \zeta$. Clearly we have

$$\mathcal{W}_q(\mathbf{X}_t, \mathbf{Y}_t) \leq \mathcal{W}_q(\mathbf{X}, \mathbf{Y}) \quad \text{for any } t \in [0, 1].$$

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- Lower bounds hold for all Lévy processes in the domain of attraction:
 - normal** case: Kantorovich's duality & the Toscani-Fourier lower bound
 - non-normal** case: extension of [BG18] for Prokhorov distance lower bound.

Where are asymptotics of $\mathcal{W}_q(\mathbf{X}^t, \mathbf{Z})$ (as $t \downarrow 0$) relevant?

- Controlling bias due to the discretisation error on a fixed time horizon [Iva18]
- Quantifying model misspecification risk [BM19]
- Analysing Euler scheme for Lévy driven SDEs, where the Lévy driver is substituted by a stable process
- Estimating semimartingale characteristics of a jumps process using high-frequency data
-

Overview of the results

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non-normal	$L(t) \lesssim \max\{\mathcal{W}_q(\mathbf{X}^t, \mathbf{Z}), \mathcal{W}_q(\mathbf{X}^{2t}, \mathbf{Z})\} \lesssim L(t)^q$, where ¹ $L(t) := t^{-1} G'(t^{-1}) /G(t^{-1})$ cannot be dominated by a non-decreasing integrable function $\ell > 0$: if $\int_0^1 \ell(t)t^{-1}dt < \infty$, then $\limsup_{t \downarrow 0} L(t)/\ell(t) = \infty$.

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Recall: $\mathbf{X}_1^t = \mathbf{X}_t / (t^{1/\alpha} G(1/t)) \xrightarrow{d} \mathbf{Z}_1$ as $t \downarrow 0$, where $G \in \text{SV}_\infty$. Assume $1 < \alpha \leq 2$.

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Fact 1: $\max \{ \mathcal{W}_1(\mathbf{X}_1^t, \mathbf{Z}_1), \mathcal{W}_1(\mathbf{X}_1^{2t}, \mathbf{Z}_1) \} \geq |1 - G(1/(2t))/G(1/t)| \mathbb{E}[|\mathbf{Z}_1|]/3, \forall \text{ small } t > 0$.

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(d) $\exists G, \tilde{G} \in \text{SV}_\infty$, such that $G(t) \sim \tilde{G}(t)$ as $t \uparrow \infty$ **and** the rates $L, \tilde{L}$ in (1) satisfy

$$0 = \liminf_{t \downarrow 0} L(t)/\tilde{L}(t) < \limsup_{t \downarrow 0} L(t)/\tilde{L}(t) = \infty.$$

Coupling general pure-jump Lévy processes $\mathbf{X}$ and $\mathbf{Y}$

Let $\Xi_{\mathbf{X}}$ and $\Xi_{\mathbf{Y}}$ be the PPPs of jumps of $\mathbf{X}$ and $\mathbf{Y}$, respectively, with corresponding compensated measures $\tilde{\Xi}_{\mathbf{X}} = \Xi_{\mathbf{X}} - \text{Leb} \otimes \nu_{\mathbf{X}}$ and $\tilde{\Xi}_{\mathbf{Y}} = \Xi_{\mathbf{Y}} - \text{Leb} \otimes \nu_{\mathbf{Y}}$.

$$\begin{aligned} D_t^{\mathbf{X}, \kappa} &= \int_{[0, t] \times (\mathbb{R}^d \setminus \{\mathbf{0}\})} \mathbb{1}_{\{|\mathbf{v}| < \kappa\}} \mathbf{v} \tilde{\Xi}_{\mathbf{X}}(ds, d\mathbf{v}), & J_t^{\mathbf{X}, \kappa} &= \int_{[0, t] \times (\mathbb{R}^d \setminus \{\mathbf{0}\})} \mathbb{1}_{\{|\mathbf{v}| \geq \kappa\}} \mathbf{v} \Xi_{\mathbf{X}}(ds, d\mathbf{v}), \\ D_t^{\mathbf{Y}, \kappa} &= \int_{[0, t] \times (\mathbb{R}^d \setminus \{\mathbf{0}\})} \mathbb{1}_{\{|\mathbf{v}| < \kappa\}} \mathbf{v} \tilde{\Xi}_{\mathbf{Y}}(ds, d\mathbf{v}), & J_t^{\mathbf{Y}, \kappa} &= \int_{[0, t] \times (\mathbb{R}^d \setminus \{\mathbf{0}\})} \mathbb{1}_{\{|\mathbf{v}| \geq \kappa\}} \mathbf{v} \Xi_{\mathbf{Y}}(ds, d\mathbf{v}). \end{aligned}$$

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Since $\mathbf{X} = D^{\mathbf{X},\kappa} + J^{\mathbf{X},\kappa}$ and $\mathbf{Y} = D^{\mathbf{Y},\kappa} + J^{\mathbf{Y},\kappa}$, we need to couple the PPPs $\Xi_{\mathbf{X}}$ and $\Xi_{\mathbf{Y}}$.

Thinning coupling

- Choose any Lévy measure μ on $\mathbb{R}^d$ that satisfies $\mu \gg \nu_{\mathbf{X}}, \nu_{\mathbf{Y}}$ with Radon-Nikodym derivatives bounded by 1, i.e. $f_{\mathbf{X}} = d\nu_{\mathbf{X}}/d\mu \leq 1$ and $f_{\mathbf{Y}} = d\nu_{\mathbf{Y}}/d\mu \leq 1$ on $\text{supp}(\mu)$.

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$$\Xi_{\mathbf{X}} = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\vartheta_n \leq f_{\mathbf{X}}(\mathbf{V}_n)\}} \delta_{(U_n, \mathbf{V}_n)} \quad \text{and} \quad \Xi_{\mathbf{Y}} = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\vartheta_n \leq f_{\mathbf{Y}}(\mathbf{V}_n)\}} \delta_{(U_n, \mathbf{V}_n)}$$

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Mapping theorem $\implies \Xi_{\mathbf{X}}$ and $\Xi_{\mathbf{Y}}$ have mean measures $\text{Leb} \otimes \nu_{\mathbf{X}}$ and $\text{Leb} \otimes \nu_{\mathbf{Y}}$, respectively.

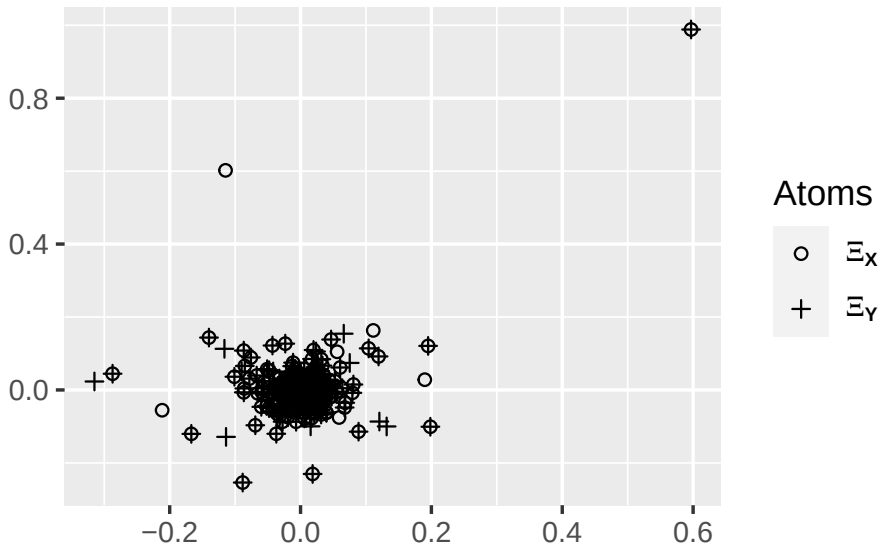


Figure: We see the atoms of Ξ_X and Ξ_Y from the **thinning coupling**.

Bounds for the thinning coupling of general Lévy processes $\mathbf{X}$ and $\mathbf{Y}$

Proposition 1

The coupling $(\mathbf{D}^{\mathbf{X},\kappa}, \mathbf{D}^{\mathbf{Y},\kappa}, \mathbf{J}^{\mathbf{X},\kappa}, \mathbf{J}^{\mathbf{Y},\kappa})$, defined by the thinning coupling, implies the following inequality, for all $q \in (0, 1]$:

$$\begin{aligned} \mathcal{W}_q(\mathbf{X}, \mathbf{Y}) \leq & |\gamma_{\mathbf{X},\kappa} - \gamma_{\mathbf{Y},\kappa}|^q + \left(4 \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \mathbb{1}_{\{|\mathbf{v}| < \kappa\}} |\mathbf{v}|^2 |f_{\mathbf{X}}(\mathbf{v}) - f_{\mathbf{Y}}(\mathbf{v})| \mu(d\mathbf{v}) \right)^{q/2} \\ & + |\Sigma_{\mathbf{X}} - \Sigma_{\mathbf{Y}}|^q + \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \mathbb{1}_{\{|\mathbf{v}| \geq \kappa\}} |\mathbf{v}|^q |f_{\mathbf{X}}(\mathbf{v}) - f_{\mathbf{Y}}(\mathbf{v})| \mu(d\mathbf{v}). \end{aligned}$$

Comonotonic coupling of $\mathbf{X}$ and $\mathbf{Y}$ with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$

Assums $\nu_{\mathbf{X}}, \nu_{\mathbf{Y}}$ have radial decomposition [LeP81, p. 282]: for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{\mathbf{0}\})$,

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where $\{\rho_{\mathbf{X}}^0(\cdot, \mathbf{v})\}_{\mathbf{v} \in \mathbb{S}^{d-1}}$ is a family of Lévy measures on $(0, \infty)$ and $\sigma_{\mathbf{X}}$ probability on $\mathbb{S}^{d-1}$.

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- Let $(\mathbf{V}_n)_{n \in \mathbb{N}}$ iid random vectors on $\mathbb{S}^{d-1}$ with law σ , independent of $(U_n, \Gamma_n)_{n \in \mathbb{N}}$

Comonotonic coupling of $\mathbf{X}$ and $\mathbf{Y}$ with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$

$$\nu_{\mathbf{X}}(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}_B(x\mathbf{v}) \rho_{\mathbf{X}}(dx, \mathbf{v}) \sigma(d\mathbf{v}), \quad \nu_{\mathbf{Y}}(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}_B(x\mathbf{v}) \rho_{\mathbf{Y}}(dx, \mathbf{v}) \sigma(d\mathbf{v}).$$

For every $\mathbf{v} \in \mathbb{S}^{d-1}$, let $u \mapsto \rho_{\mathbf{X}}^{\leftarrow}(u, \mathbf{v})$ (resp. $u \mapsto \rho_{\mathbf{Y}}^{\leftarrow}(u, \mathbf{v})$) be the generalised inverse of $x \mapsto \rho_{\mathbf{X}}([x, \infty), \mathbf{v})$ (resp. $x \mapsto \rho_{\mathbf{Y}}([x, \infty), \mathbf{v})$).

Construction inspired by [Ros01, LeP81]:

- Let $(\Gamma_n)_{n \in \mathbb{N}}$ RW with standard exponential increments, independent of iid- $U[0, 1]$ sequence $(U_n)_{n \in \mathbb{N}}$
- Let $(\mathbf{V}_n)_{n \in \mathbb{N}}$ iid random vectors on $\mathbb{S}^{d-1}$ with law σ , independent of $(U_n, \Gamma_n)_{n \in \mathbb{N}}$
- Define PPP Ξ on $[0, 1] \times (0, \infty) \times \mathbb{S}^{d-1}$ with mean measure $\text{Leb} \otimes \text{Leb} \otimes \sigma$

$$\Xi := \sum_{n \in \mathbb{N}} \delta_{(U_n, \Gamma_n, \mathbf{V}_n)}, \quad \widetilde{\Xi}(ds, dx, d\mathbf{v}) = \Xi(ds, dx, d\mathbf{v}) - ds \otimes dx \otimes \sigma(d\mathbf{v}).$$

Comonotonic coupling of $\mathbf{X}$ and $\mathbf{Y}$ with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$

For any $\epsilon \in (0, \infty)$ (and even $\epsilon = \infty$ when $\mathbf{X}$ and $\mathbf{Y}$ both have jumps of finite variation), we can write the small-jump components of $\mathbf{X}$ and $\mathbf{Y}$ as

$$M_t^{\mathbf{X}} = \int_{[0,t] \times (\epsilon, \infty) \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) \widetilde{\Xi}(ds, dx, d\mathbf{v}),$$

$$M_t^{\mathbf{Y}} = \int_{[0,t] \times (\epsilon, \infty) \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v}) \widetilde{\Xi}(ds, dx, d\mathbf{v}).$$

Comonotonic coupling of $\mathbf{X}$ and $\mathbf{Y}$ with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$

For any $\epsilon \in (0, \infty)$ (and even $\epsilon = \infty$ when $\mathbf{X}$ and $\mathbf{Y}$ both have jumps of finite variation), we can write the small-jump components of $\mathbf{X}$ and $\mathbf{Y}$ as

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The big-jump components can similarly be written as

$$\mathbf{L}_t^{\mathbf{X}} = \int_{[0,t] \times (0, \epsilon] \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) \Xi(ds, dx, d\mathbf{v}), \quad \mathbf{L}_t^{\mathbf{Y}} = \int_{[0,t] \times (0, \epsilon] \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v}) \Xi(ds, dx, d\mathbf{v}).$$

Comonotonic coupling of $\mathbf{X}$ and $\mathbf{Y}$ with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$

For any $\epsilon \in (0, \infty)$ (and even $\epsilon = \infty$ when $\mathbf{X}$ and $\mathbf{Y}$ both have jumps of finite variation), we can write the small-jump components of $\mathbf{X}$ and $\mathbf{Y}$ as

$$\begin{aligned} M_t^{\mathbf{X}} &= \int_{[0,t] \times (\epsilon, \infty) \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) \widetilde{\Xi}(ds, dx, d\mathbf{v}), \\ M_t^{\mathbf{Y}} &= \int_{[0,t] \times (\epsilon, \infty) \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v}) \widetilde{\Xi}(ds, dx, d\mathbf{v}). \end{aligned}$$

The big-jump components can similarly be written as

$$L_t^{\mathbf{X}} = \int_{[0,t] \times (0, \epsilon] \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) \Xi(ds, dx, d\mathbf{v}), \quad L_t^{\mathbf{Y}} = \int_{[0,t] \times (0, \epsilon] \times \mathbb{S}^{d-1}} \mathbf{v} \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v}) \Xi(ds, dx, d\mathbf{v}).$$

Then we have $\mathbf{X}_t = \varpi_{\mathbf{X}} t + \Sigma_{\mathbf{X}} \mathbf{B}_t^{\mathbf{X}} + M_t^{\mathbf{X}} + L_t^{\mathbf{X}}$ and $\mathbf{Y}_t = \varpi_{\mathbf{Y}} t + \Sigma_{\mathbf{Y}} \mathbf{B}_t^{\mathbf{Y}} + M_t^{\mathbf{Y}} + L_t^{\mathbf{Y}}$ for some constants $\varpi_{\mathbf{X}}, \varpi_{\mathbf{Y}} \in \mathbb{R}^d$ and all $t \in [0, 1]$.

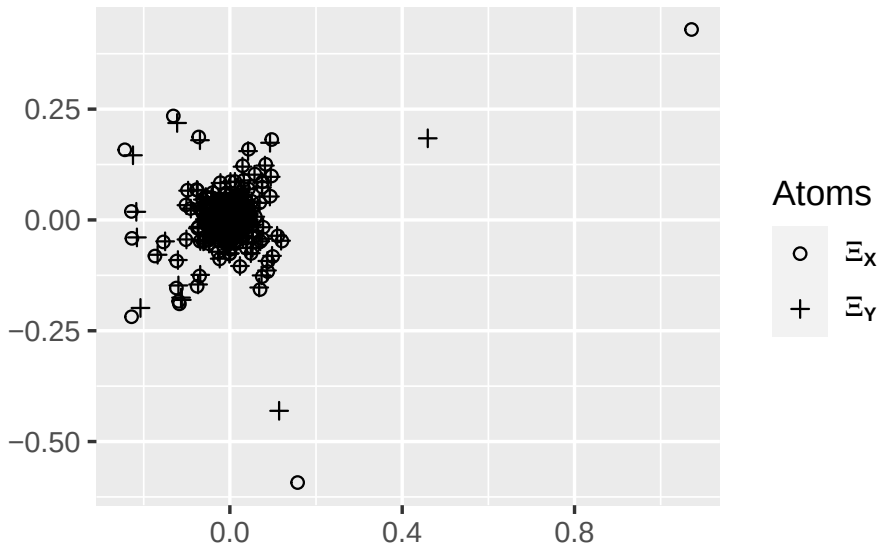


Figure: We see the atoms of Ξ_X and Ξ_Y from the **comonotonic coupling**.

Bound for the comonotonic coupling

Proposition 2

For any $\epsilon \in (0, \infty)$ (and even $\epsilon = \infty$ when $\mathbf{X}$ and $\mathbf{Y}$ are of finite variation), the coupling $(\mathbf{M}^{\mathbf{X}}, \mathbf{M}^{\mathbf{Y}}, \mathbf{L}^{\mathbf{X}}, \mathbf{L}^{\mathbf{Y}})$ implies the following inequality, for any $q \in (0, 1]$:

$$\begin{aligned} \mathcal{W}_q(\mathbf{X}, \mathbf{Y}) \leq & |\varpi_{\mathbf{X}} - \varpi_{\mathbf{Y}}|^q + \left(4 \int_{(\epsilon, \infty) \times \mathbb{S}^{d-1}} (\rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) - \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v}))^2 dx \otimes \sigma(d\mathbf{v}) \right)^{q/2} \\ & + |\Sigma_{\mathbf{X}} - \Sigma_{\mathbf{Y}}|^q + \int_{(0, \epsilon] \times \mathbb{S}^{d-1}} |\rho_{\mathbf{X}}^{\leftarrow}(x, \mathbf{v}) - \rho_{\mathbf{Y}}^{\leftarrow}(x, \mathbf{v})|^q dx \otimes \sigma(d\mathbf{v}). \end{aligned} \quad (2)$$

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Thank you for your attention! Paper is available on arXiv.

Domain of attraction to a Brownian motion

Theorem 5.1

Let Σ be a symmetric non-negative definite matrix on $\mathbb{R}^{d \times d}$ and define $\mathbf{X}^t = ((\Sigma \mathbf{B}_{st} + \mathbf{Y}_{st})/\sqrt{t})_{s \in [0,1]}$ for $t \in (0, 1]$ where $(\mathbf{B}_t)_{t \in [0,1]}$ is a standard Brownian motion on $\mathbb{R}^d$ independent of the tempered α -stable Lévy process $\mathbf{Y}$.

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(a) Suppose that $\mathbf{Y}$ is of infinite variation, that $\alpha \in [1, 2)$ and fix any $\epsilon > 0$ such that $\alpha + \epsilon \leq 2$. Then,

$$\mathcal{W}_1(\mathbf{X}^t, \Sigma \mathbf{B}) = \mathcal{O}(t^{1/(\alpha+\epsilon)-1/2}), \quad \text{as } t \downarrow 0.$$

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(b) There exist some $C_* > 0$, such that $\mathcal{W}_1(\mathbf{X}_1^t, \Sigma \mathbf{B}_1) \geq C_* t^{1-\alpha/2}$ for all sufficiently small $t > 0$.