

Subexponential lower bounds for f -ergodic Markov processes

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Part I: Results

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Motivation: ergodic elliptic diffusion in $\mathbb{R}^n$

Let

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

and denote $\Sigma = \sigma\sigma^\top$.

Let $\alpha, \beta, \gamma \in (0, \infty)$ and $\ell \in [0, 2)$, with $m_c := 2 + (2\alpha - \gamma)/\beta > 2 - \ell$.

Assume as $|x| \rightarrow \infty$:

$$\langle b(x), x/|x| \rangle \sim -\alpha|x|^{\ell-1}, \quad \langle \Sigma(x)x/|x|, x/|x| \rangle \sim \beta|x|^\ell, \quad \text{Tr}(\Sigma(x)) \sim \gamma|x|^\ell$$

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Then [FR05] X is ergodic with stationary measure π and

$$\|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}} \leq Ct^{-m_c/(2-\ell)+1} \quad \text{for } t \in [1, \infty).$$

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How about the lower bound?

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Let $V : \mathcal{X} \rightarrow [1, \infty)$ continuous, $\phi : [1, \infty) \rightarrow [1, \infty)$ increasing, concave, $A \in \mathcal{B}(\mathcal{X})$ petite set and $b \in \mathbb{R}_+$, such that

$V(X) + \int_0^\cdot \phi \circ V(X_u) du - b \int_0^\cdot \mathbb{1}\{X_u \in A\} du$ is a supermartingale,

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(a) X Harris recurrent with invariant π and $\int_{\mathcal{X}} \phi \circ V(x) \pi(dx) < \infty$

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How about the **lower bounds**?

L-drift condition $\mathbf{L}(\mathbf{V}, \varphi, \Psi)$

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(i) $\varphi : (0, 1] \rightarrow \mathbb{R}_+$ continuous, $r\varphi(1/r) \downarrow 0$ as $r \uparrow \infty$ and

$$1/V(X) - \int_0^\cdot \varphi(1/V(X_u)) du - b \int_0^\cdot \mathbb{1}\{V(X_u) \leq \ell_0\} du \quad \text{supermart.}$$

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(ii) For $\Psi : [1, \infty) \rightarrow [1, \infty)$ submultip., any $\ell > \ell_0$, $\exists C_\ell > 0$ s.t.

$\mathbb{P}_x(T^{(r)} < S_{(\ell)}) \geq C_\ell / \Psi(r)$ for all $\ell + 1 < V(x) < r < \infty$ where

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Remark.

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Lower bounds on convergence rates

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Theorem (Lower Bounds)

Let $\mathbf{L}(\mathbf{V}, \varphi, \Psi)$ hold.

(a) For any $\varepsilon, q \in (0, 1)$, there exists $c_{\varepsilon,q} > 0$ such that

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(ii) Let $\mathbf{a} : [1, \infty) \rightarrow \mathbb{R}_+$ with $\mathbf{a}(r) \leq c_{\varepsilon,q}/L_{\varepsilon,q}(\mathbf{h}^{-1}(r))$ be such that $\mathbf{A}(r) := r\mathbf{a}(r) \uparrow \infty$ as $r \rightarrow \infty$.

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Then, for all $x \in \mathcal{X}$ and $t \in [1, \infty)$, we have

$$(\mathbf{a} \circ \mathbf{A}^{-1} \circ (2\mathbf{v}))(x, t)/2 \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}}.$$

Elliptic diffusion X in $\mathbb{R}^n$: **polynomial** tails of π

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X satisfies $\mathbf{L}(\mathbf{V}, \varphi, \Psi)$ for any $\varepsilon > 0$ with (for large $x \in \mathbb{R}^n$ and $r \geq 1$):

$$\mathbf{V}(x) = |x|^{m_c - \varepsilon}, \quad \varphi(1/r) = r^{(\ell-3)/(m_c - \varepsilon)}/C_0, \quad \Psi(r) = r^{1+2\varepsilon/(m_c - \varepsilon)}$$

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Recall

$$\pi(\mathbf{V} \geq r) \geq \frac{C_{\varepsilon, q}}{L_{\varepsilon, q}(r)}, \quad \text{where } L_{\varepsilon, q}(r) := r\varphi(1/r)\Psi(2r/(1-q))(\log \log r)^\varepsilon.$$

Hence, by **Theorem LB(a)**, we get

$$\pi(\{|x| \geq r\}) \geq c_\pi / r^{m_c + \ell - 2 + 2\varepsilon} \quad \text{for all } r \in [1, \infty).$$

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Hence, for $k \in [0, \ell + (2\alpha - \gamma)/\beta)$ and $\eta_k := (m_c - k)/(2 - \ell) - 1 > 0$,
all $x \in \mathbb{R}^n$ and small $\varepsilon > 0$, we have

$$c_{k,\varepsilon}/t^{\eta_k+\varepsilon} \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{1+|x|^k} \quad \forall t \in [1, \infty).$$

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Our **lower** bounds match the well-known **upper** [FR05] bounds: $\forall x \in \mathbb{R}^n$

$$c_{k,\varepsilon}/t^{\eta_k+\varepsilon} \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{1+|x|^k} \leq C_{k,\varepsilon}/t^{\eta_k-\varepsilon} \quad \forall t \in [1, \infty).$$

Special case: Langevin diffusions in $\mathbb{R}^n$

$\tilde{\pi} : \mathbb{R}^n \rightarrow (0, \infty)$ proportional to density of π

$$dX_t = \nabla(\log \tilde{\pi})(X_t)dt + dB_t.$$

Assume $\langle \nabla(\log \tilde{\pi})(x), x/|x| \rangle \sim -\alpha/|x|$ as $|x| \rightarrow \infty$ with $\alpha > n/2$.

Hence $\ell = 0, \gamma = n, \beta = 1$ and $\eta_0 = (2\alpha - n)/2$. By **Theorem LB(b)**

$$c_{\text{TV}}/t^{(2\alpha-n)/2+\varepsilon} \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}} \leq C_{\text{TV}}/t^{(2\alpha-n)/2-\varepsilon}$$

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Alternatively:

$$\lim_{t \rightarrow \infty} \frac{\log \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}}}{\log t} = -(2\alpha - n)/2$$

Elliptic diffusions: **stretched exponential** ergodicity

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Assume: for $p \in (0, 1]$ and $\ell \in [0, 2p)$, we have

$$-\alpha_L \leq \langle b(x), x/|x| \rangle / |x|^{\ell-p} \leq -\alpha_U \quad \& \quad \beta_L \leq \langle \Sigma(x)x/|x|, x/|x| \rangle / |x|^\ell \leq \beta_U.$$

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$\mathbf{h}(r) = \exp(r^{1-p})$ ensures superlinear growth of $\mathbb{E}_x[\mathbf{h} \circ \mathbf{V}(X_t)]$ in t .

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Lower bounds in **Theorem LB** match well-known **upper** [DFG09] bounds:

$$\begin{aligned} c_\pi / \exp(b_\pi r^{1-p}) &\leq \pi(\{x \in \mathbb{R}^n : |x| \geq r\}) \leq C_\pi / \exp(B_\pi r^{1-p}), \\ c / \exp(bt^{(1-p)/(1+p-\ell)}) &\leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}} \leq C / \exp(Bt^{(1-p)/(1+p-\ell)}). \end{aligned}$$

Elliptic diffusions: **stretched exponential** ergodicity

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Assume: for $p \in (0, 1]$ and $\ell \in [0, 2p]$, we have

$$-\alpha_L \leq \langle b(x), x/|x| \rangle / |x|^{\ell-p} \leq -\alpha_U \quad \& \quad \beta_L \leq \langle \Sigma(x)x/|x|, x/|x| \rangle / |x|^\ell \leq \beta_U.$$

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Exponential ergodicity: analogous bounds hold for $p = 0 = \ell$.

Lévy driven Ornstein-Uhlenbeck process

Let $\mu \in (0, \infty)$ and L be a pure-jump Lévy process in $\mathbb{R}$. Let X satisfy

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$$0 < \liminf_{r \rightarrow \infty} (\log r)^{m_c} \nu([r, \infty)) \leq \limsup_{r \rightarrow \infty} (\log r)^{m_c} \nu([r, \infty)) < \infty$$

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Lower bounds from **Theorem LB** match **upper** bounds [FR05]: $\forall \varepsilon > 0$,

$$c_\pi / (\log r)^{m_c - 1 + \varepsilon} \leq \pi([r, \infty)) \leq C_\pi / (\log r)^{m_c - 1 - \varepsilon},$$

$$c/t^{m_c - 1 + \varepsilon} \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}} \leq C/t^{m_c - 1 - \varepsilon}.$$

Concluding remarks

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Thank you for your attention!

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