

Subexponential lower bounds for f -ergodic Markov processes

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Part II: Proofs

EPSRC

Engineering and Physical Sciences
Research Council

EP/W006227/1 & EP/V009478/1

L-drift condition $\mathbf{L}(\mathbf{V}, \varphi, \Psi)$

X ergodic Markov, positive Harris recurrent with invariant measure π .

$V : \mathcal{X} \rightarrow [1, \infty)$ cts, $\limsup_{t \rightarrow \infty} V(X_t) = \infty$ and $b, \ell_0 \in [0, \infty)$

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Remark.

(ii) holds if $\Psi \circ V(X) + b \int_0^\cdot \mathbb{1}\{V(X_u) \leq \ell_0\} du$ is a submartingale.

Lower bounds on convergence rates

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Theorem (Lower Bounds)

Let $\mathbf{L}(\mathbf{V}, \varphi, \Psi)$ hold and recall

$$L_{\varepsilon, q}(r) := r\varphi(1/r)\Psi(2r/(1-q))(\log \log r)^\varepsilon, \quad r \in [1, \infty).$$

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Then, for all $x \in \mathcal{X}$ and $t \in [1, \infty)$, we have

$$(\mathbf{a} \circ \mathbf{A}^{-1} \circ (2\mathbf{v}))(x, t)/2 \leq \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}}.$$

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Skeleton of the proof of **Theorem LB** in three Steps:

$L(V, \varphi, \Psi) \xRightarrow{\text{Step I}}$ lower bound on $\mathbb{P}_x \left(\int_0^{\tau_\delta(D)} h \circ V(X_s) ds > r \right)$ & every **petite** set is bounded

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Step I(a): return times to bounded sets

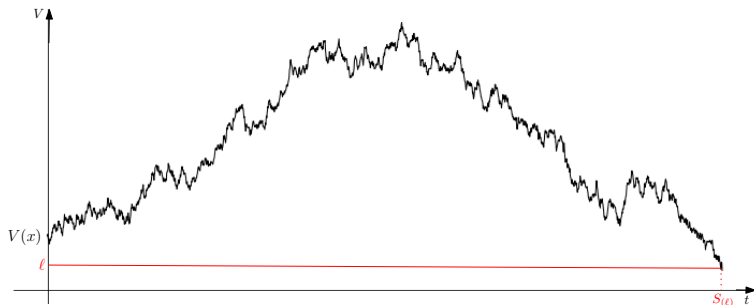
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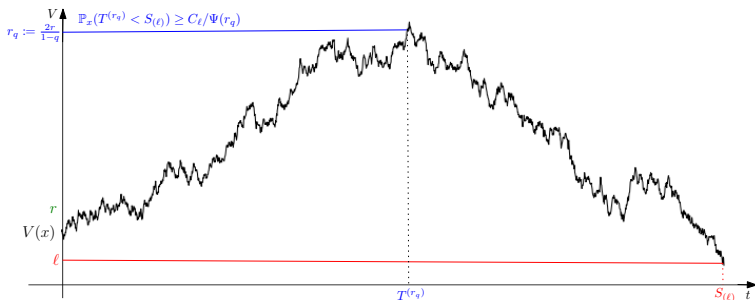
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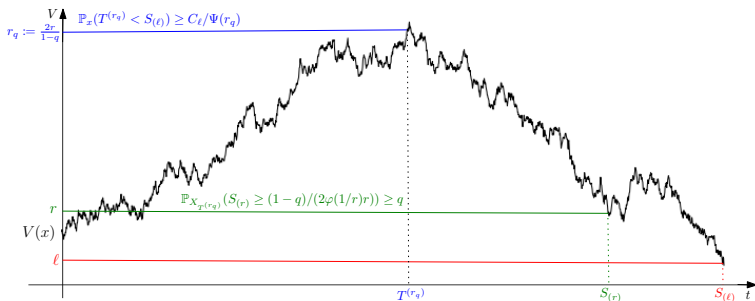
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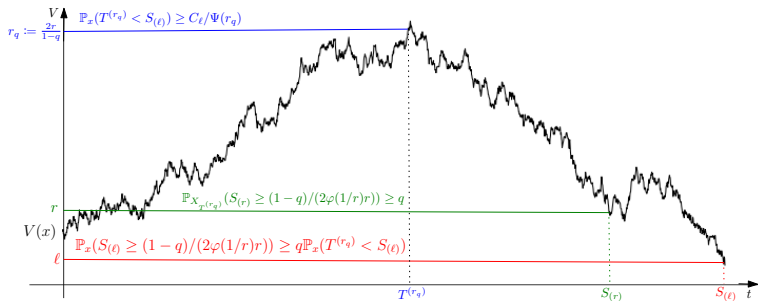
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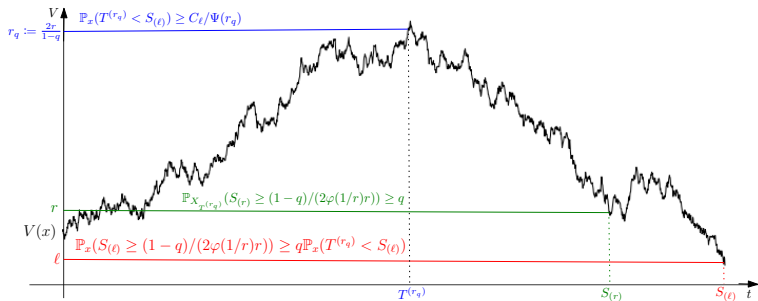
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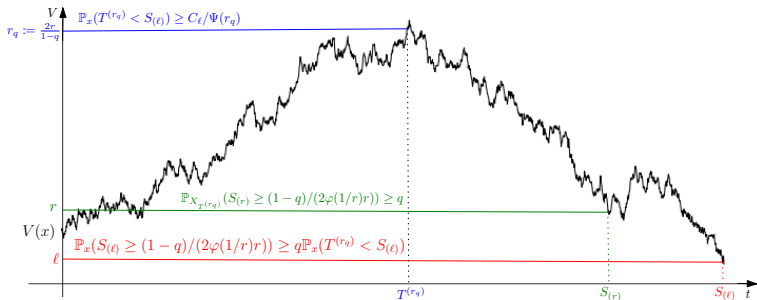
Thus, $\forall x \in \mathcal{X}$, we have $\mathbb{P}_x(S_{(\ell)} \geq t) \geq \begin{cases} q \frac{C_\ell}{\Psi(2G_1(t)/(1-q))}, & V(x) < \frac{2G_1(t)}{1-q}, \\ q, & V(x) \geq \frac{2G_1(t)}{1-q}, \end{cases}$

where G_1 is the inverse of $r \mapsto \frac{1-q}{2r\varphi(1/r)}$ (Recall $q \in (0, 1)$ arbitrary!)

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More generally,
$$\mathbb{P}_x \left(\int_0^{S_{(\ell)}} h \circ \mathbf{V}(X_s) ds \geq t \right) \geq q \frac{C_\ell}{\Psi(2G_h(t)/(1-q))},$$

for **any** non-decreasing h , where G_h is the inverse of $r \mapsto \frac{(1-q)h(r)}{2r\varphi(1/r)}$.

Step I(b): petite sets are bounded

B is a **petite set** if $\exists$ independent time τ_a and positive measure ν , s.t.

$$\inf_{x \in B} \mathbb{P}_x(X_{\tau_a} \in \cdot) \geq \nu(\cdot) \quad \text{on } \mathcal{B}(\mathcal{X}).$$

If $\tau_a \equiv t_0 > 0$ is constant, B is a **small set**.

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Then $\exists \ell > 1$ s.t. $\nu(V \leq \ell) > 0$. Recall $S_{(\ell)} := \inf\{t > 0 : V(X_t) \leq \ell\}$.

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By **Step I(a)**, for every $x \in \{V > 2G_1(t_0)/(1 - q)\}$, we have

$$\mathbb{P}_x(S_{(\ell)} \geq t_0) > q := 1 - \nu(V \leq \ell)$$

where G_1 is the inverse of $r \mapsto (1 - q)/(2r\varphi(1/r))$. Hence

$$B \subset \{V \leq 2G_1(t_0)/(1 - q)\}.$$

Step II From return times to invariant measures

Meyn and Tweedie [MT93]: Let $H : \mathcal{X} \rightarrow [1, \infty)$ be measurable. Then:

$$\forall D \text{ petite, } \exists \delta > 0 \text{ s.t. } \sup_{x \in D} \mathbb{E}_x \left[\int_0^{\tau_\delta(D)} H(X_s) ds \right] = \infty \stackrel{(\star)}{\Rightarrow} \int_{\mathcal{X}} H(x) \pi(dx) = \infty$$

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Integral test $(\star)$, inequality $(\star\star)$ and a technical construction imply:

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We have increasing $\mathbf{h}$ such that

$$\mathbb{E}_x[\mathbf{h} \circ V(X_t)] \leq v(x, t) \quad \& \quad \mathbf{a}(r) \leq c_{\varepsilon, q} / L_{\varepsilon, q}(\mathbf{h}^{-1}(r)) \leq \pi(\mathbf{h} \circ V \geq r)$$

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By Hairer's argument [Hai21],

$$\begin{aligned} \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}} &\geq \pi(\{\mathbf{h} \circ V \geq r\}) - \mathbb{P}_x(\mathbf{h} \circ V(X_t) \geq r) \\ &\geq \mathbf{a}(r) - v(x, t)/r, \end{aligned}$$

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The f -variation distance alternative takes the following form:

$$\|\pi(\cdot) - \mathbb{P}_x(X_t \in \cdot)\|_f \geq \int_{\{G \geq r\}} f(x)\pi(\mathrm{d}x) - \mathbb{E}_x[f(X_t)\mathbb{1}\{G(X_t) \geq r\}],$$

where $G := (\mathbf{h} \circ V)/f$ is assumed to be increasing.

Concluding remarks

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Proofs are **probabilistic** and **self contained**, except the dependence on the integral test for invariant measures [MT93, Thm 1.2(b)].

- [Hai21] M. Hairer. *Lecture notes on Convergence of Markov processes*. available at <https://www.hairer.org/notes/Convergence.pdf>. May 2021.
- [MT93] S.P. Meyn and R.L. Tweedie. “Generalized resolvents and Harris recurrence of Markov processes”. *Doeblin and modern probability (Blaubeuren, 1991)*. Vol. 149. Contemp. Math. Amer. Math. Soc., Providence, RI, 1993, pp. 227–250.