

Superdiffusive limits for Bessel-driven stochastic kinetics

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Stochastic drift model

Bessel process X of *stochastic dimension*

(Y, B) adapted in $\mathbb{R}_+ \times \mathbb{R}$, B Brownian Motion, X in $\mathbb{R}_+ := [0, \infty)$,

$$X_t = X_0 + \int_0^t \frac{f(s, Y_s)}{X_s} ds + B_t, \quad X_0 > 0.$$

$(\mathbf{A}_f)$ Continuous $f : \mathbb{R}_+^2 \rightarrow (0, \infty)$ with $f(t, y) \sim \rho t^\gamma y^\alpha$ as $t, y \rightarrow \infty$ for some constants $\rho \in (0, \infty)$ and $\alpha, \gamma \in \mathbb{R}_+$, such that $\alpha + \gamma > 0$.

Let Y be a squared Bessel process of positive “dimension” $\delta \in (0, \infty)$:

$$\text{BESQ}^\delta(y): \quad dY_t = \delta dt + 2Y_t^{1/2} dW_t, \text{ for } Y_0 = y \in \mathbb{R}_+.$$

X exhibits *anomalous diffusion*.

Path of $dX_t = \frac{\sqrt{Y_t}}{X_t} dt + dB_t$, where $Y \sim \text{BESQ}^1(0)$ (here $f(t, y) = \sqrt{y}$)

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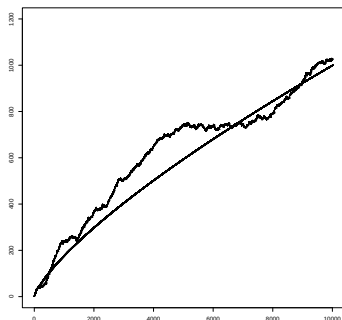


Figure: Simulated path $t \mapsto X_t$ on $[0, T]$, with $T = 10^5$, and graph of $t \mapsto t^{3/4}$.

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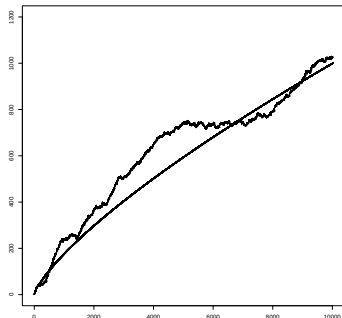


Figure: Simulated path $t \mapsto X_t$ on $[0, T]$, with $T = 10^5$, and graph of $t \mapsto t^{3/4}$.

X appears to be **superdiffusive**!

Main result

BESQ $^\delta(y)$: $dY_t = \delta dt + 2Y_t^{1/2}dW_t$, for $Y_0 = y \in \mathbb{R}_+$, $\delta \in (0, \infty)$

Theorem (Y-Bes)

Assume $f(t, y) \sim \rho t^\gamma y^\alpha$ as $t, y \rightarrow \infty$ with $\gamma + \alpha > 0$. Adapted process (X, Y, B) , $Y \sim \text{BESQ}^\delta(y)$ with $\delta > 0$ and X in $\mathbb{R}_+$ satisfies

$$X_t = X_0 + \int_0^t \frac{f(s, Y_s)}{X_s} ds + B_t, \quad X_0 > 0.$$

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(II) There is a distributional limit: let $\tilde{Y} \sim \text{BESQ}^\delta(0)$, then

$$t^{-(1+\gamma+\alpha)/2} X_t \xrightarrow{d} \left(2\rho \int_0^1 s^\gamma \tilde{Y}_s^\alpha ds \right)^{1/2}, \text{ as } t \rightarrow \infty.$$

Remarks

$$X_t = X_0 + \int_0^t (f(s, Y_s)/X_s) ds + B_t \text{ and } f(t, y) \sim \rho t^\gamma y^\alpha \text{ as } t, y \rightarrow \infty$$

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Log-scale convergence $\lim_{t \rightarrow \infty} \frac{\log X_t}{\log t} = \frac{1+\gamma+\alpha}{2} \iff$ for small $\varepsilon > 0$, a.s.
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Arbitrary dependence (Y, B) is allowed: as $dY_t = \delta dt + 2|Y_t|^{1/2}dW_t$,
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If $\gamma + \alpha = 0$ (i.e., $\alpha = \gamma = 0$ and hence $f \sim \text{const}$), then X close to Bessel proc. of dimension $1 + 2\rho > 1$ (i.e. diffusive) with random distributional limit of $t^{-1/2}X_t$. Limit in Thm (Y-Bes)(II) (deterministic for $\alpha = 0$) cannot hold, i.e. $\gamma + \alpha > 0$ is necessary!

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If Y *ergodic* with stationary measure π and $f(s, y) = h(y)$, then X
behaves as Bessel process of dimension $\pi(h)$, i.e. X likely diffusive.
Interesting case is when Y null-recurrent (or transient).

What should Y satisfy?

$$X_t = X_0 + \int_0^t (f(s, Y_s)/X_s) ds + B_t \text{ and } f(t, y) \sim \rho t^\gamma y^\alpha \text{ as } t, y \rightarrow \infty$$

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($\mathbf{A}_Y$) Let $\alpha + \gamma > 0$ and $A_t := \int_0^t s^\gamma Y_s^\alpha ds$ satisfies the following:

- (a) For a rv $\tilde{A}$ in $\mathbb{R}_+$, $A_t/t^{1+\gamma+\alpha} \xrightarrow{d} \tilde{A}$ as $t \rightarrow \infty$.
- (b) For every $\varepsilon > 0$, we have $\lim_{t \rightarrow \infty} t^\varepsilon A_t/t^{1+\gamma+\alpha} = \infty$, a.s.
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Theorem (Y -gen)

Let Y satisfy **(A_Y)** and the adapted process (X, Y, B) be as above. Then

$$\lim_{t \rightarrow \infty} \frac{\log X_t}{\log t} = \frac{1 + \gamma + \alpha}{2}, \text{ a.s.,}$$

and, as $t \rightarrow \infty$, for $\tilde{A}$ the random variable in **(A_Y)(c)**, we have

$$t^{-(1+\gamma+\alpha)/2} X_t \xrightarrow{d} (2\rho\tilde{A})^{1/2}.$$

Proof of Theorem (Y -gen): $A_t := \int_0^t s^\gamma Y_s^\alpha ds$

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For any $t \in \mathbb{R}_+$, Itô's formula yields $X_t^2 = X_0^2 + U_t + t + M_t$, where

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Weak limit: let $\beta := \alpha + \gamma > 0$. For any $\varepsilon > 0$, $(\mathbf{A}_Y)(c)$, Doob inequality and BC implies $\sup_{0 \leq s \leq t} |M_s| < t^{1+\beta/2+\varepsilon}$ for all large t . By $(\mathbf{A}_Y)(a)$,

$$X_t^2/t^{1+\beta} = (X_0^2 + t + M_t)/t^{1+\beta} + U_t/t^{1+\beta} \xrightarrow{d} 2\rho\tilde{A}, \text{ as } t \rightarrow \infty.$$

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a.s. limit: for any $\varepsilon > 0$ a.s. for all large t we have

$$(\mathbf{A}_Y)(b) \implies U_t/t^{1+\beta} > t^{-\varepsilon}; \quad (\mathbf{A}_Y)(c) \implies U_t/t^{1+\beta} < t^\varepsilon$$

Hence,

$$\begin{aligned} 1 + \beta - \varepsilon &\leq \frac{\log X_t^2}{\log t} = 1 + \beta + \log((X_0^2 + t + M_t)/t^{1+\beta} + U_t/t^{1+\beta}) / \log t \\ &\leq 1 + \beta + \varepsilon \end{aligned}$$

Proof of Theorem (Y-Bes): $A_t := \int_0^t s^\gamma Y_s^\alpha ds$

If $Y \sim \text{BESQ}^\delta(y)$, $\delta > 0$, $y \in \mathbb{R}_+$, satisfies $(\mathbf{A}_Y)$: $\alpha + \gamma > 0$ and for $A_t = \int_0^t s^\gamma Y_s^\alpha ds$ the following hold:

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Recall (SP): $\forall s > 0$, we have $Y_s \sim s\tilde{Y}_1$, where $\tilde{Y} \sim \text{BESQ}^\delta(y/s)$

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$(\mathbf{A}_Y)$ (b): for $\delta \in (0, 2)$, work needed for a.s. $A_t > t^{1+\gamma+\alpha-\varepsilon}$ for large t .
Balance functional of excursion of Y from 1 to 0 with its duration

(BP): $Y \sim \bar{Y} + Y'$, $\bar{Y} \sim \text{BESQ}^1(0)$, $Y' \sim \text{BESQ}^{\delta-1}(y)$ independent

$$(\text{BP}) \implies A_t \stackrel{d}{=} \int_0^t s^\gamma (\bar{Y}_s + Y'_s)^\alpha ds \geq \int_0^t s^\gamma \bar{Y}_s^\alpha ds \geq t^{(1+\alpha+\gamma)-\varepsilon}$$

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Thank you for your attention!!