

Fast exact simulation of the first-passage event of a subordinator

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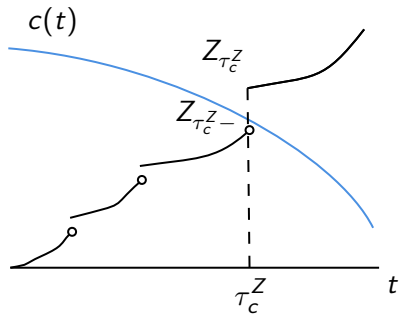
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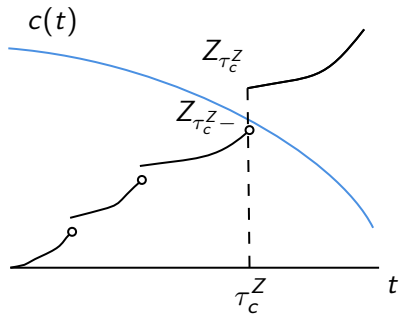
Engineering and Physical Sciences
Research Council

June 4, 2023

EP/W006227/1 & EP/V009478/1



Z general subordinator, $c : [0, \infty) \rightarrow [0, \infty)$ non-increasing absolutely continuous function



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Problem: find and analyse **efficient exact** simulation algorithm for

$$(\tau_{\mathbf{c}}^Z, Z_{\tau_{\mathbf{c}}^Z-}, Z_{\tau_{\mathbf{c}}^Z})$$

Plan for the talk

Part 1. **Our exact simulation **Algorithm 1** and its complexity?** (this video)

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Part 2. **Proof, motivation and numerical examples** (next video)

General class of subordinators

Z driftless infinite activity subordinator, $Z_0 = 0$, with Lévy measure ν_Z

$$\nu_Z = \nu_{r,q} + \lambda_r, \quad \text{where } \nu_{r,q}(dx) := \mathbb{1}_{\{0 < x \leq r\}} \vartheta e^{-qx} x^{-\alpha-1} dx, \quad x \in (0, \infty), \quad (1.1)$$

for some parameters $\vartheta \in (0, \infty)$, $q \geq 0$, $\alpha \in (0, 1)$, $r \in (0, \infty]$ and Lévy measure λ_r on $(0, \infty)$ with finite total mass $\Lambda_r := \lambda_r(0, \infty) \in [0, \infty)$.

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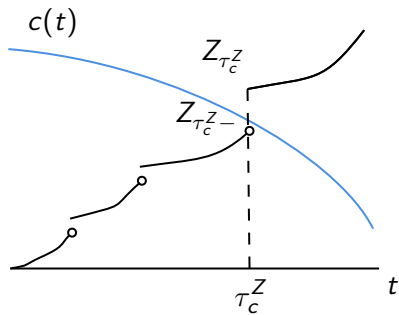
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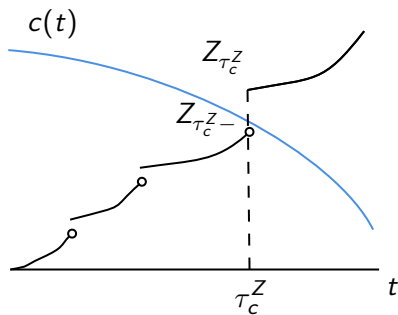
This is a general class of subordinators: (1.1) holds if e.g. $\nu_Z(dx) = \nu_Z(x)dx$ has a **density near zero** satisfying

$$\vartheta := \lim_{x \downarrow 0} \nu_Z(x) x^{\alpha+1} \in (0, \infty) \quad \& \quad \limsup_{x \downarrow 0} |\nu_Z(x) x^{\alpha+1} - \vartheta|/x < \infty.$$

The problem: first passage across a non-increasing curve



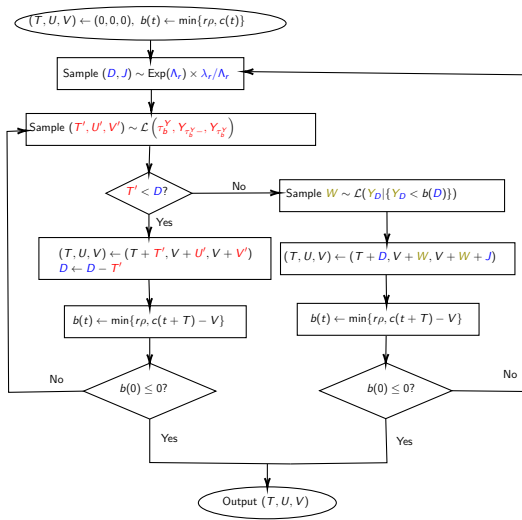
The problem: first passage across a non-increasing curve



Recall:

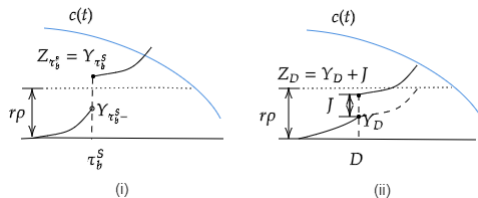
$Z = Y + Q$: Y truncated tempered stable; Q compound Poisson with Lévy measure λ_r

Flowchart: Algorithm 1



$Z = Y + Q$: Y truncated tempered stable; Q compound Poisson with Lévy measure λ_r

Local behaviour of Algorithm 1

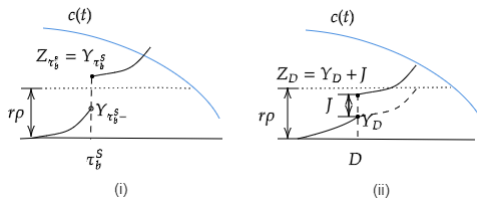


Barrier function $b(t) = \min\{r\rho, c(t)\}$, r truncation parameter, $\rho = 1/2$

(i) Y crosses b before Q jumps

(ii) Q jumps before Y crosses b

Local behaviour of Algorithm 1



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(i) Y crosses b before Q jumps

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In both cases, the barrier and the triplet are updated and the algorithm checks whether the original barrier $c(t)$ has been crossed. If not (resp. yes), the procedure is repeated on the updated barrier and starting point (resp. the updated triplet is returned).

Algorithm 1 Simulation of χ_c^Z of a subordinator with Lévy measure in (1.1)

Require: Parameters $\alpha, \rho \in (0, 1)$, $\vartheta \in (0, \infty)$, $r \in (0, \infty]$, $q \in [0, \infty)$, finite measure λ_r in (1.1) and non-increasing function c

- 1: Set $(T, U, V) \leftarrow (0, 0, 0)$, $b(t) = \min\{c(t), r\rho\}$, $\theta = \vartheta\Gamma(1 - \alpha)/\alpha$ and generate $D \sim \text{Exp}(\Lambda_r)$
 - 2: **repeat** generate $(T', U', V') \sim \mathcal{L}(\tau_b^Y, Y_{\tau_b^Y-}, Y_{\tau_b^Y})$ via **Algorithm 4**
 - 3: **if** $D > T'$ **then** ▷ Y crosses b before Q jumps
 - 4: Set $(T, U, V) \leftarrow (T + T', V + U', V + V')$
 - 5: Set $D \leftarrow D - T'$ and $b(t) = \min\{c(t + T) - V, r\rho\}$
 - 6: **else if** $D \leq T'$ **then** ▷ Q jumps before Y crosses b
 - 7: Generate $W \sim \mathcal{L}(Y_D | \{Y_D < b(D)\})$ via **Algorithm 2** and $J \sim \lambda_r/\Lambda_r$
 - 8: Set $(T, U, V) \leftarrow (T + D, V + W, V + W + J)$
 - 9: Generate $D \sim \text{Exp}(\Lambda_r)$ and set $b(t) = \min\{c(t + T) - V, r\rho\}$
 - 10: **end if**
 - 11: **until** $b(0) \leq 0$
 - 12: **return** (T, U, V)
-

Key step (I)

Algorithm 2 Simulation from the law of $S_t | \{S_t < s\}$ where S_t tempered stable ($q > 0$)

Require: Parameters $\alpha \in (0, 1)$, $\theta, q, t, s > 0$

- 1: **repeat** generate $E \sim \text{Exp}(1)$
 - 2: Generate W from the law of $S_t | \{S_t < s\}$ via Algorithm 3 where S stable ($q = 0$)
 - 3: **until** $E > qW$
 - 4: **return** W
-

Algorithm 3 Simulation from the law of $S_t | \{S_t < s\}$ where S_t stable ($q = 0$)

Require: Parameters $\alpha \in (0, 1)$, $\theta, t, s > 0$

- 1: Set $\beta = \alpha/(1 - \alpha)$ and generate $E' \sim \text{Exp}(1)$
 - 2: Generate U from the density proportional to $\mathbb{1}_{(0,1)}(u) \exp(-\sigma_\alpha(u)(\theta t)^{\beta+1}s^{-\beta})$ via Devroye's algorithm [2]
 - 3: **return** $(s^{-\beta} + (\theta t)^{-\beta-1}E'/\sigma_\alpha(U))^{-1/\beta}$
-

Key steps(II)

Assumption (FPTS-Alg)

Assume that the running time of the first-passage tempered-stable algorithm (FPTS-Alg) for generating a sample from the law of $(\tau_c^S, S_{\tau_c^S-}, S_{\tau_c^S})$ has a finite exponential moment. Denote an upper bound on the expected running time of FPTS-Alg by $T_0 = T_0(\alpha, \theta, q, c)$.

Algorithm 4 Simulation of $(\tau_b^Y, Y_{\tau_b^Y-}, Y_{\tau_b^Y})$ for a truncated tempered stable subordinator Y and a function $b : [0, \infty) \rightarrow [0, \infty)$ bounded by the truncation level r

Require: Parameters $\alpha, \rho \in (0, 1)$, $\vartheta \in (0, \infty)$, $r \in (0, \infty]$, $q \in [0, \infty)$ and $b \leq r$

- 1: **repeat** generate $(T, U, V) \sim \mathcal{L}(\tau_b^S, S_{\tau_b^S-}, S_{\tau_b^S})$ via FPTS-Alg
 - 2: **until** $V - U \leq r$ $\triangleright T$ is the first crossing time of Y over b
 - 3: **return** (T, U, V)
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Algorithm 5 Simulation of $(\tau_b^Y, Y_{\tau_b^Y-}, Y_{\tau_b^Y})$ for a truncated tempered stable subordinator Y and a function $b : [0, \infty) \rightarrow [0, \infty)$ bounded by the truncation level r

Require: Parameters $\alpha, \rho \in (0, 1)$, $\vartheta \in (0, \infty)$, $r \in (0, \infty]$, $q \in [0, \infty)$ and $b \leq r$

- 1: **repeat** generate $(T, U, V) \sim \mathcal{L}(\tau_b^S, S_{\tau_b^S-}, S_{\tau_b^S})$ via FPTS-Alg
 - 2: **until** $V - U \leq r$ $\triangleright T$ is the first crossing time of Y over b
 - 3: **return** (T, U, V)
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A possible choice of FPTS-Alg is [1, Alg. 1] with a bound T_0 given in [1, Thm 2.1].

Main theorem

Theorem 1

The running time of Algorithm 1 has moments of all orders if FPTS-Alg does. The expected running time of Algorithm 1 is bounded above by

$$\kappa_1 \left(\frac{T_0}{1 - \rho^\alpha} + e^{q \min\{c_0, r\rho\}} \right) \left(\frac{\Lambda_r}{\psi_0 + \vartheta \Upsilon(1/(1 + c_0), r_0, q)} + \frac{c_0}{r\rho} \right), \quad (2.1)$$

where T_0 is expected running time of FPTS-Alg, $\psi_0 := \int_{(0, \infty)} (1 - e^{-x/(1+c_0)}) \lambda_{r_0}(dx)$ and the constant κ_1 does not depend on $\alpha \in (0, 1)$, $\vartheta \in (0, \infty)$, $c_0 := c(0) \in (0, \infty)$, $\Lambda_r = \lambda_r(0, \infty) < \infty$, $\rho \in (0, 1)$, $r \in (0, r_0]$ or $q \in [0, \infty)$.

Υ is explicit and satisfies the following limits:

$$\lim_{q \rightarrow \infty} \Upsilon(u, r, q)q = u/2 \quad \& \quad \lim_{u \rightarrow 0} \Upsilon(u, r, q)/u = (q^{-1} \wedge r \wedge 1)/2.$$

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It is natural to choose the truncation at $r = \min\{r_0, 2\alpha/q\}$.

When FPTS-Alg is [1, Alg. 1] and $\rho = 1/2$, $r = \min\{2\alpha/q, r_0\}$, we get

Corollary 1

Expected running time $\mathcal{T}(\alpha, \vartheta, q, c_0)$ of Algorithm 1 is bounded by

$$\kappa_2 \left((1 - \alpha)^{-3} + |\log \alpha| + \log N \right) \left(\frac{\alpha \Lambda_{r_0} + \vartheta \mathbb{1}_{\{2\alpha/q < r_0\}} q^\alpha}{\psi_0 + \vartheta \Upsilon(1/(1 + c_0), r_0, q)} + c_0 q \right) / \alpha^3$$

where the constant κ_2 does not depend on any of the quantities: $\alpha \in (0, 1)$, $\vartheta \in (0, \infty)$, $c_0 = c(0) \in (0, \infty)$, $q \in (0, \infty)$ or $N \in \mathbb{N}$.

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Thus

$$\mathcal{T}(\alpha, \vartheta, q, c_0) = \begin{cases} \mathcal{O}(|\log \alpha| \alpha^{-3}), & \alpha \rightarrow 0, \\ \mathcal{O}((1 - \alpha)^{-3}), & \alpha \rightarrow 1, \\ \mathcal{O}(q(1 + \mathbb{1}_{\{\Lambda_{r_0}=0\}} q^\alpha)), & q \rightarrow \infty, \end{cases} \quad \mathcal{T}(\alpha, \vartheta, q, c_0) = \begin{cases} \mathcal{O}(c_0), & c_0 \rightarrow \infty, \\ \mathcal{O}(1), & \vartheta \rightarrow 0, \\ \mathcal{O}(1), & \vartheta \rightarrow \infty. \end{cases}$$

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Tempered stable bound ([1, Cor. 2.2]): $\mathcal{O}(q)$ as $q \rightarrow \infty$ and $\mathcal{O}(|\log \alpha| \alpha^{-2})$ as $\alpha \rightarrow 0$;

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Tempered stable bound ([1, Cor. 2.2]): $\mathcal{O}(q)$ as $q \rightarrow \infty$ and $\mathcal{O}(|\log \alpha| \alpha^{-2})$ as $\alpha \rightarrow 0$; in other cases, there is no deterioration.

References & resources

- [1] Jorge Ignacio González Cázares, Feng Lin, and Aleksandar Mijatović. *Fast exact simulation of the first passage of a tempered stable subordinator across a non-increasing function*. 2023. [arXiv: 2303.11964 \[math.PR\]](#).
- [2] Luc Devroye. “A note on generating random variables with log-concave densities”. *Statistics & Probability Letters* 82.5 (2012), pp. 1035–1039.

See also:

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GitHub with implementation in **Python** and **Julia** of **Algorithm 1** above, using **Python** and **Julia** of [1, Algs 1 & 2] (**GitHub** links embedded in pdf)