

Fast exact simulation of the first-passage event of a subordinator

Part II

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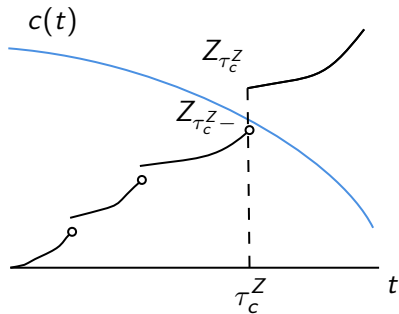
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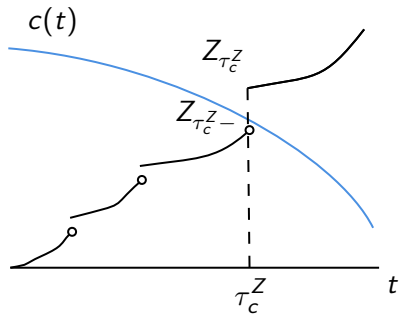
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Problem: find and analyse **efficient exact** simulation algorithm for

$$(\tau_{\mathbf{c}}^Z, Z_{\tau_{\mathbf{c}}^Z-}, Z_{\tau_{\mathbf{c}}^Z})$$

General class of subordinators

Z driftless infinite activity subordinator, $Z_0 = 0$, with Lévy measure ν_Z

$$\nu_Z = \nu_{r,q} + \lambda_r, \quad \text{where } \nu_{r,q}(dx) := \mathbb{1}_{\{0 < x \leq r\}} \vartheta e^{-qx} x^{-\alpha-1} dx, \quad x \in (0, \infty), \quad (0.1)$$

for some parameters $\vartheta \in (0, \infty)$, $q \geq 0$, $\alpha \in (0, 1)$, $r \in (0, \infty]$ and Lévy measure λ_r on $(0, \infty)$ with finite total mass $\Lambda_r := \lambda_r(0, \infty) \in [0, \infty)$.

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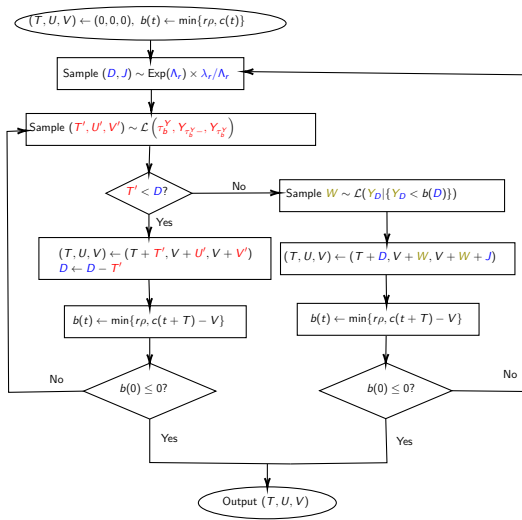
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This is a general class of subordinators: (0.1) holds if e.g. $\nu_Z(dx) = \nu_Z(x)dx$ has a **density near zero** satisfying

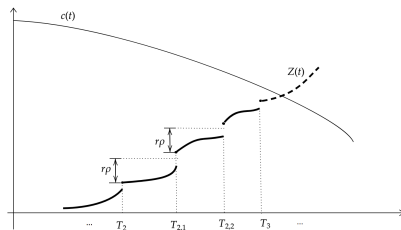
$$\vartheta := \lim_{x \downarrow 0} \nu_Z(x) x^{\alpha+1} \in (0, \infty) \quad \& \quad \limsup_{x \downarrow 0} |\nu_Z(x) x^{\alpha+1} - \vartheta|/x < \infty.$$

Flowchart: Algorithm 1



$Z = Y + Q$: Y truncated tempered stable; Q compound Poisson with Lévy measure λ_r

Proof: consider **Algorithm 1** on the entire time interval $[0, \tau_c^Z]$:



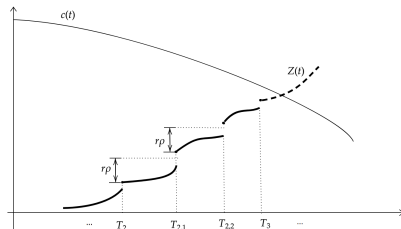
$K \geq 0$ jumps of the compound Poisson process Q in $[0, \tau_c^Z]$;

$T_1, \dots, T_K$ jump times of Q in $[0, \tau_c^Z]$;

Crossing times of Y during $[T_i, T_{i+1}]$:

$$T_{i,j+1} = \inf \{t > T_{i,j} : Y_t - Y_{T_{i,j}} > \min\{c(t) - Z_{T_{i,j}}, r\rho\}\}, \quad j = 0, 1, \dots$$

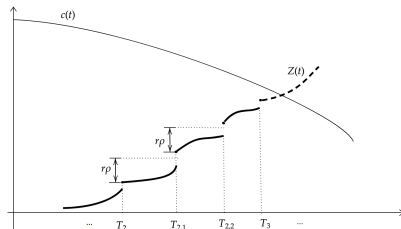
Ideas behind the proof of our main complexity theorem



We have to **control** running time of **Algorithm 1** during intervals:

$$0 = T_0 < T_{0,1} \dots < T_1 < T_{1,1} < \dots < T_{1,L_1} < T_K < T_{K,1} < \dots < T_{K,L_K} = \tau_c^Z,$$

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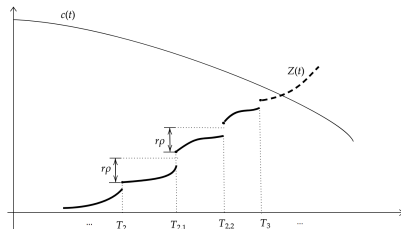


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- Running times are in this sequence are **dependent**.

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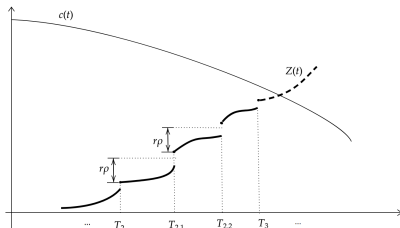


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- Running times in this sequence are **dependent**.
- Sampling $\mathcal{L}(Y_D | \{Y_D < b(D)\})$, via a log-concave density [2], has uniformly bounded expected running time.
- The simulation of the triplet $(\tau_b^Y, Y_{\tau_b^Y-}, Y_{\tau_b^Y})$ is based on [1, Alg. 1] for the tempered stable first-passage event.

Kolmogorov-Smirnov test for Algorithm 1

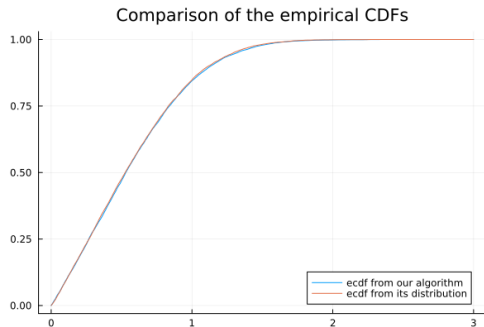
Let Z stable with Lévy measure $x^{-1.6}dx = \mathbb{1}\{0 < x < 1\}e^{-x}x^{-1.6}dx + \lambda_1(dx)$ and $\lambda_1(dx) = \mathbb{1}\{0 < x < 1\}(1 - e^{-x})x^{-1.6}dx + \mathbb{1}\{x > 1\}x^{-1.6}dx$.

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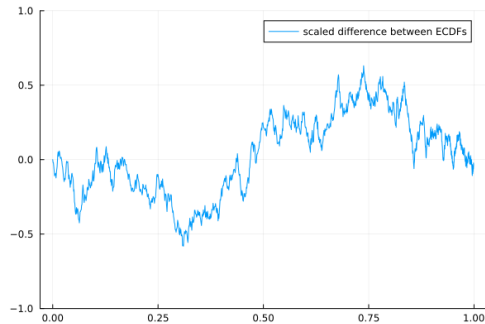
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(a) The ECDF from the algorithm and the one from its distribution have good agreement.



(b) The scaled difference between the ECDFs with $n = 10^5$ samples, resembles a Brownian bridge (passing Kolmogorov-Smirnov test)

Numerical solutions of fractional partial differential equations (FPDEs)

A_x generator of Feller process $\{X_s^x\}_{s \geq 0}$, $\phi \in \text{Dom}(A_x)$, fix $a < b$, consider FPDE:

$$\begin{aligned} (-{}_t D_a + A_x)u(t, x) &= 0, & (t, x) &\in (a, b] \times \mathbb{R}^d, \\ u(a, x) &= \phi(x), & x &\in \mathbb{R}^d, \quad \text{where } -{}_t D_a \text{ is a} \end{aligned} \tag{1.1}$$

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$$-{}_t D_a f(t) := - \int_0^{t-a} (f(t-s) - f(t)) \nu(ds) + (f(a) - f(t)) \int_{t-a}^{\infty} \nu(ds).$$

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$$u(t, x) = \mathbb{E} [\phi(X_{T_t}^x)], \quad \text{yielding a Monte Carlo estimator} \quad u_n(t, x) = \frac{1}{n} \sum_{k=1}^n \phi(X_{T_t^k}^{x,k}).$$

Example: Ornstein–Uhlenbeck process

Let $\phi(x) = x_1 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$; Feller process X^x satisfying SDE

$$dX_s = -\mu X_s ds + \gamma dW_s, \quad X_0^x = x, \quad \text{where } \mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (1.2)$$

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Generalised Caputo-type operator $-_t D_a$ given by Lévy measure

$$\nu(ds) = \frac{1}{-\Gamma(-0.65)} \mathbb{1}\{0 < s < 1\} e^{-s} s^{-1.65} ds + \mathbb{1}\{s \geq 1\} s^{-5} ds.$$

Driftless subordinator $t - Y^{a,t}$ has Lévy measure in (0.1) with $\alpha = 0.65$, $q = 1$, the truncation parameter $r = 1$ and the finite activity component $\lambda_1(ds) = \mathbb{1}\{s \geq 1\} s^{-5} ds$.

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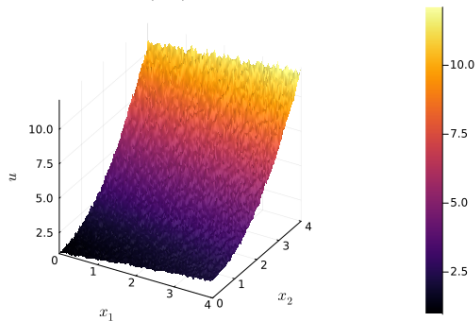
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Sample $T_t = \inf\{s > 0 : t - Y_s^{a,t} > t - a\}$ using our main Algorithm 1 and $X_{T_t}^\times$:

$$u_n(t, x) = \frac{1}{n} \sum_{k=1}^n \phi(X_{T_t^k}^{\times,k}) \quad \text{for the solution } u(t, x) = \mathbb{E} [\phi(X_{T_t}^\times)] \text{ of FPDE (1.1).}$$

Solutions of fractional partial differential equations (FPDEs)

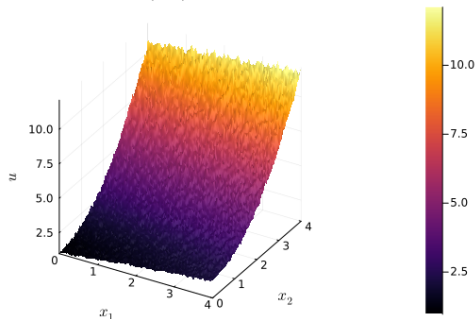
Solution $u(t, x)$ of the FPDE



Using **Algorithm 1** to simulate $T_t = \inf\{s > 0 : t - Y_s^{a,t} > t - a\}$, we get Monte Carlo approximation for the solution of FPDE (1.1) at time $t - a = 5$.

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Many generalisations are possible!

References & resources

- [1] Jorge Ignacio González Cázares, Feng Lin, and Aleksandar Mijatović. *Fast exact simulation of the first passage of a tempered stable subordinator across a non-increasing function*. 2023. [arXiv: 2303.11964 \[math.PR\]](#).
- [2] Luc Devroye. “A note on generating random variables with log-concave densities”. *Statistics & Probability Letters* 82.5 (2012), pp. 1035–1039.

See also:

<https://sites.google.com/view/aleks-mijatovic/research> (or comments below the video)
for arXiv link to the preprint

<https://sites.google.com/view/aleks-mijatovic/videos> for this presentation

GitHub with implementation in **Python** and **Julia** of **Algorithm 1** above, using **Python** and **Julia** of [1, Algs 1 & 2] (**GitHub** links embedded in pdf)