

Fast exact simulation of the first passage event of a tempered stable subordinator (Part 2)

Jorge González Cázares



Feng Lin



Aleksandar Mijatović



EPSRC

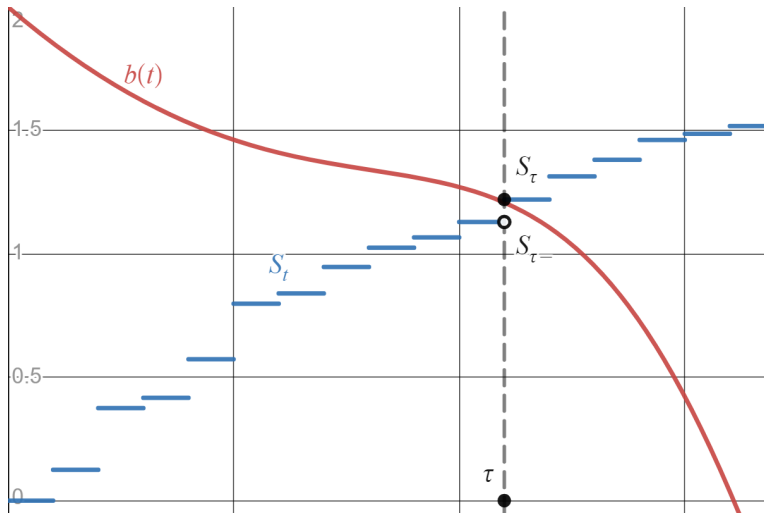
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The problem: first passage across a non-increasing curve



Basic setting

For any stability index $\alpha \in (0, 1)$, intensity $\theta \in (0, \infty)$ and tempering parameter $q \in \mathbb{R}_+$, let $S = (S_t)_{t \in \mathbb{R}_+}$ be, under $\mathbb{P}_q$ (with expectation operator $\mathbb{E}_q$), a driftless tempered stable subordinator started at 0 with Laplace transform and Lévy measure

$$\begin{aligned}\mathbb{E}_q[e^{-uS_1}] &= \exp((q^\alpha - (u + q)^\alpha)\theta), & u \geq 0 \\ \nu_q(dx) &= w_\alpha^{-1} \theta e^{-qx} x^{-\alpha-1} dx, & x > 0,\end{aligned}$$

where $w_\alpha = \Gamma(1 - \alpha)/\alpha$.

Main result

Let $b : [0, \infty) \rightarrow [0, \infty)$, with $b(0) \in (0, \infty)$, be a non-increasing absolutely continuous function. Let S be a tempered stable subordinator with parameters $\alpha \in (0, 1)$, $\theta > 0$ and $q \geq 0$. We develop and analyse an exact simulation algorithm below for the random element

$$(\tau_b, S_{\tau_b-}, S_{\tau_b}), \quad \text{where } \tau_b := \inf\{t > 0 : S_t > b(t)\},$$

with expected running time bounded by

$$\kappa(1 + qb(0)/\alpha)((1 - \alpha)^{-3} + |\log \alpha| + \log N)/\alpha, \quad (1.1)$$

where κ depends on neither $\alpha \in (0, 1)$, $\theta \in (0, \infty)$ nor $q \in [0, \infty)$, and N specifies the required number of precision bits of the output.

Solutions of fractional partial differential equations (FPDEs)

Let $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function that vanishes at infinity and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$. For $a < b$, consider the following fractional partial differential equations:

$$\begin{aligned} (-_t D_a + A_x)u(t, x) &= -g(t, x), & (t, x) &\in (a, b] \times \mathbb{R}^d, \\ u(a, x) &= \phi(x), & x &\in \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where A_x is a generator of a Feller semigroup on $C_\infty(\mathbb{R}^d)$ acting on x , $\phi \in \text{Dom}(A_x)$, the operator $-_t D_a$ is a generalised differential operator of Caputo type of order less than 1 acting on the time variable $t \in [a, b]$.

Solutions of fractional partial differential equations (FPDEs)

The solution u of the FPDE problem in (2.1) exists and its stochastic representation is

$$u(t, x) = \mathbb{E} \left[\phi(X_{T_t}^x) + \int_0^{T_t} g(Y_s^{a,t}, X_s^x) ds \right]$$

where $\{X_s^x\}_{s \geq 0}$ is the stochastic process generated by A_x and $T_t = \inf\{s > 0, Y_s^{a,t} < a\}$ where $\{Y_s^{a,t}\}_{s \geq 0}$ is the decreasing $[a, b]$ -valued process started at $t \in [a, b]$ generated by $-_t D_a$. Note that T_t is the first passage time of the subordinator $(t - Y_s^{a,t})_{s \in \mathbb{R}_+}$ over the constant level $t - a$.

Solutions of fractional partial differential equations (FPDEs)

In the special case $g \equiv 0$, $\phi(x) = x^2$, $A_x = (x^2/2)\Delta$ (where Δ is the Laplacian on $\mathbb{R}$) and the subordinator is tempered stable, we can simulate the Monte Carlo estimator

$$u_n(t, x) = \frac{1}{n} \sum_{k=1}^n \phi(X_{T_t^k}^{x,k}) \quad (2.2)$$

of the solution u of the FPDE in (2.1).

Solutions of fractional partial differential equations (FPDEs)

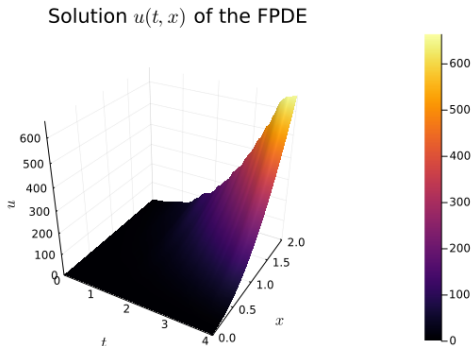


Figure: Estimated values of $u(t, x)$ using $n = 10^4$ samples. Here $\phi(x) = x^2$, T_t has the law of the first passage time of a tempered stable subordinator with parameters $\alpha = 0.4$, $\theta = 1$, $q = 1$ crossing level $b(t) \equiv 1$ and $X = (X_t)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion $X_t = x \exp(B_t - t/2)$ with $X_0 = x > 0$.

What about naive random walk approximation?

- It is tempting to simulate a random walk approximation (i.e. the skeleton) of the subordinator $(t - Y_s^{a,t})_{s \in \mathbb{R}_+}$ on $\{0, h, 2h, \dots\}$ and approximate T_t with the first time $T_t^{(h)}$ the random walk crosses level $t - a$.

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- Unlike $u_n(t, x)$, the estimator

$$u_n^{(h)}(t, x) = \frac{1}{n} \sum_{k=1}^n \phi\left(X_{T_t^{(h)}, k}^{x, k}\right), \quad (t, x) \in \mathbb{R}_+^2,$$

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- For the L^2 -error to be $\leq \epsilon^2$, the bias and variance need to be $\leq \epsilon$ and $\leq \epsilon^2$.
- The total complexities of $u_n^{(h)}(t, x)$ and $u_n(t, x)$ are of order n/h and n , respectively. Thus, the improvement in computational complexity of $u_n(t, x)$ over $u_n^{(h)}(t, x)$ is proportional to $1/h \propto \epsilon^{-1}x^2e^{t^\alpha}$.

Extension to processes of bounded variation

- Let Z be a Lévy process with non-positive drift, such that its Lévy measure satisfies

$$\nu(\mathbb{R}_+) = \infty, \quad \int_{-\infty}^{\infty} (x \wedge 1) \nu(dx) < \infty.$$

Z can be decomposed as $Z^+ - Z^-$, where Z^+ and Z^- are independent subordinators with Levy measures ν^+ and ν^- , respectively, and Z^+ has no drift.

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- Given a constant $c > 0$ and let $\tau_c^Z = \inf\{t > 0 : Z_t > c\}$. If we know how to sample the first passage event of Z^+ and the increment of Z^- , we can sample the first passage event of Z .

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- The idea is to sample the first passage of Z^+ at time τ_1^+ , compute $Z_{\tau_1^+}^-$, and determine if it is also the first passage of Z . If it is, the algorithm stops, otherwise, we sample the next first passage of $Z_t^+ - Z_{\tau_1^+}^-$ and so on.

Extension to processes of bounded variation

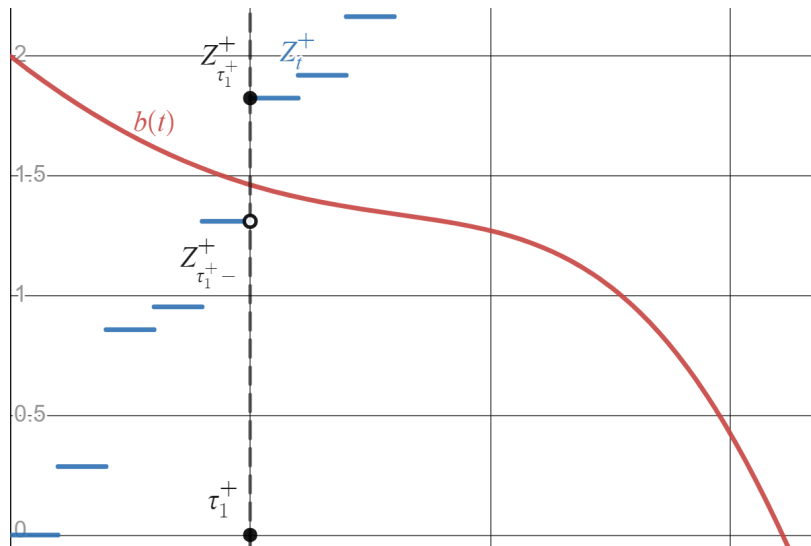


Figure: Sketch of simulating the first passage event of bounded variation process

Extension to processes of bounded variation

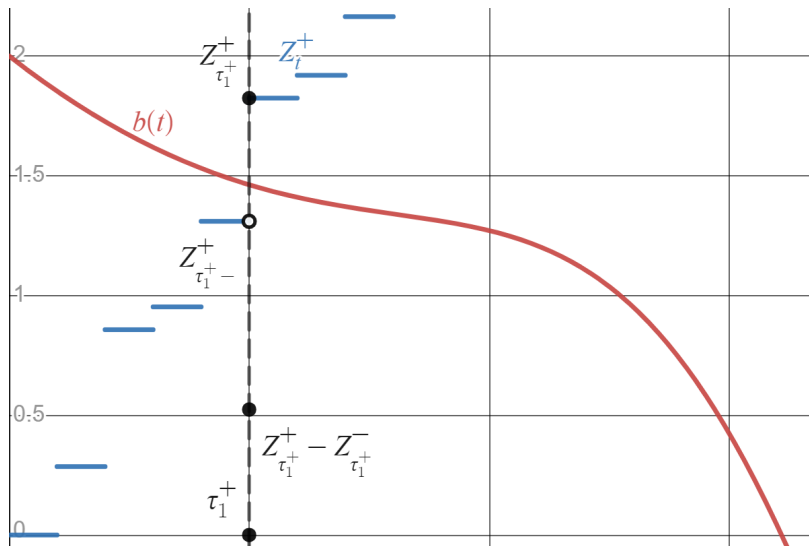


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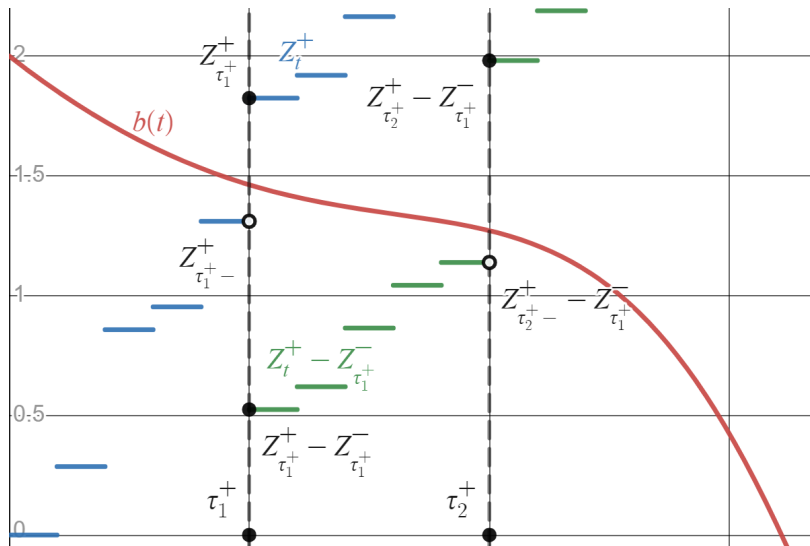


Figure: Sketch of simulating the first passage event of bounded variation process

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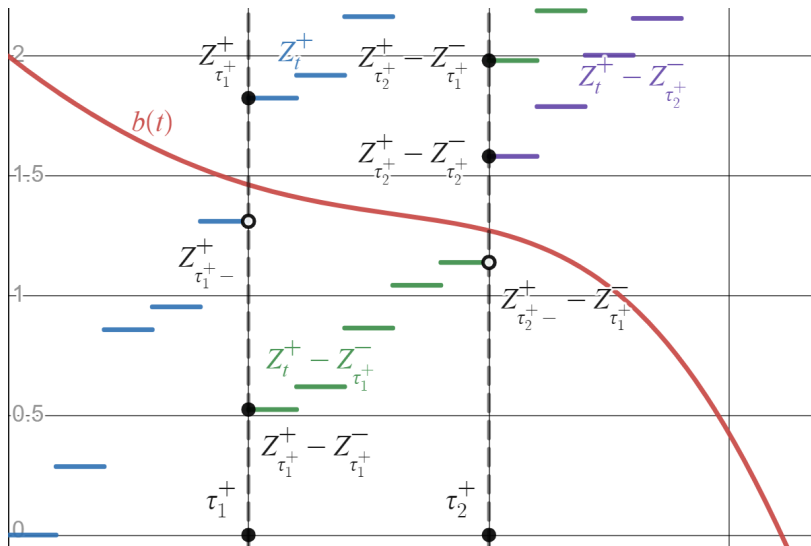


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Pricing barrier options

Let X denote a bounded variation process. Then $R_t := R_0 e^{X_t}$ models the risky asset for some initial value $R_0 > 0$ and all $t > 0$. Consider the payoff of the up-and-out barrier call option function

$$\begin{aligned} g(T, R) &= \max\{R_T - K, 0\} \mathbb{1}\left\{\sup_{t \in [0, T]} R_t < M\right\} \\ &= \max\{R_0 e^{X_T} - K, 0\} \mathbb{1}\left\{\sup_{t \in [0, T]} X_t < \log(M/R_0)\right\} \\ &= \max\{R_0 e^{X_T} - K, 0\} \mathbb{1}\{\tau_{\log(M/R_0)} > T\} \\ &= \max\{R_0 e^{X_{\tau_*}} - K, 0\} \mathbb{1}\{\tau_* = T\}, \end{aligned}$$

where $\tau_* := \min\{\tau_{\log(M/R_0)}, T\}$.

Pricing barrier options

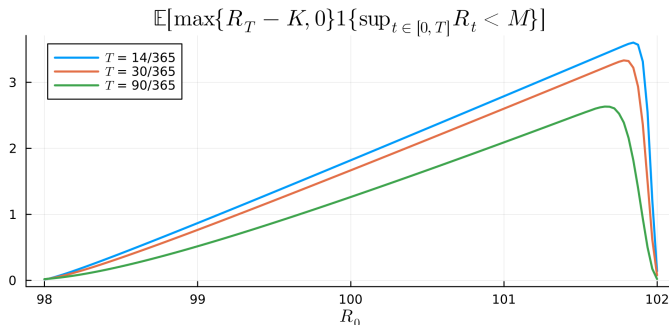


Figure: Estimated values of $g(T, R) = \max\{R_T - K, 0\} \mathbb{1}\{\sup_{t \in [0, T]} R_t < M\}$ using $N = 10^4$ samples with time horizon $T = 14/365, 30/365$ and $90/365$. We set $K = 98, M = 102$. $X = X^+ - X^-$ where the calibrated parameters of X^+ and X^- are $(\alpha^+, \theta^+, q^+) = (0.66, 0.1305, 6.5022)$ and $(\alpha^-, \theta^-, q^-) = (0.66, 0.0615, 3.3088)$, respectively, from the USD/JPY currency pair.