

Brownian motion with asymptotically normal reflection in unbounded domains: from transience to stability

Aleksandar Mijatović
Department of Statistics



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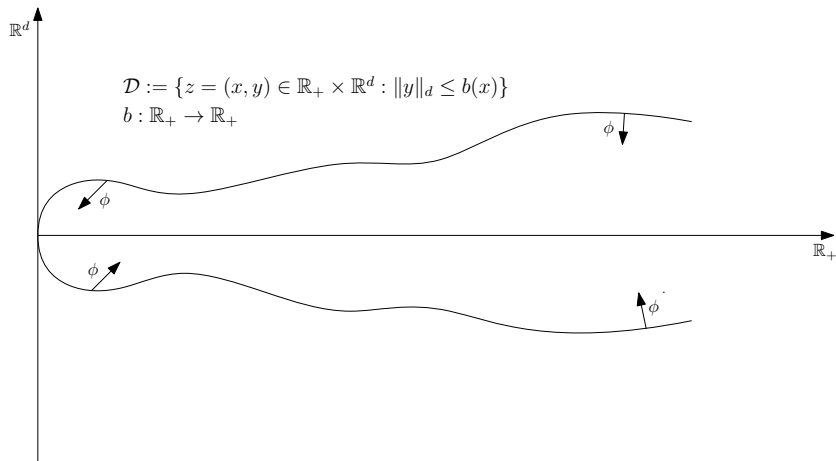
Joint work with
Miha Brešar and Andrew Wade

EPSRC

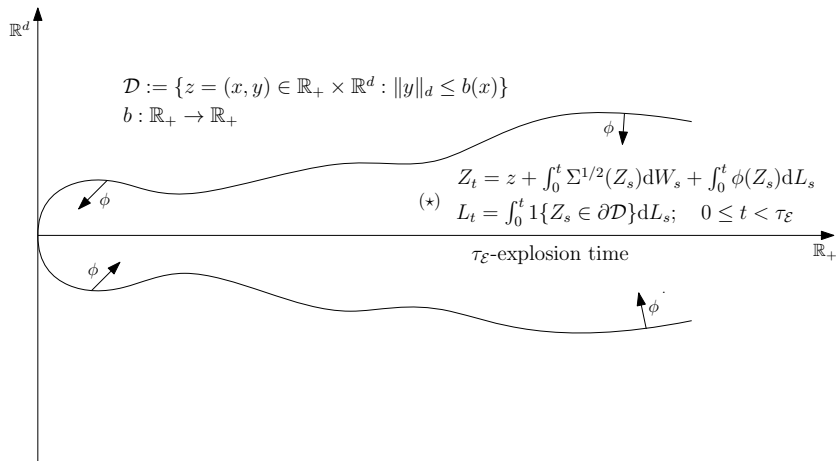
Engineering and Physical Sciences
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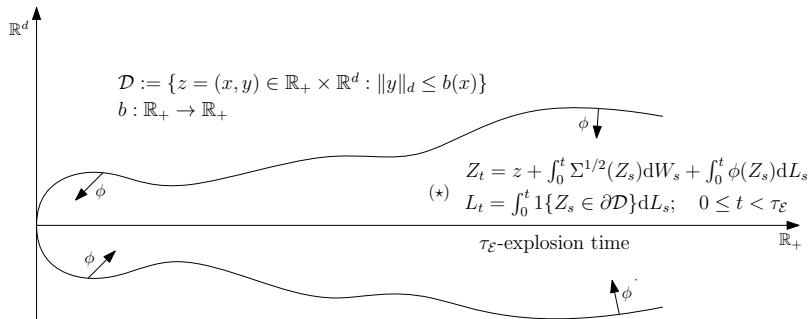
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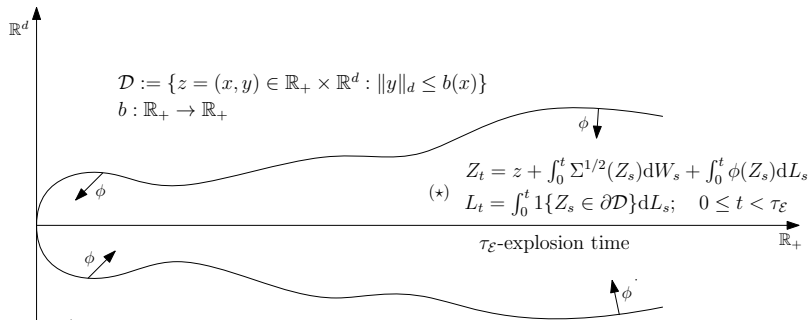
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(C1) Σ is bounded, Lipschitz and uniformly elliptic: $\langle \Sigma(z)v, v \rangle \geq \delta \|v\|_{d+1}^2$.

(V1) $\phi : \partial\mathcal{D} \rightarrow \mathbb{R}^{d+1}$ C^2 vector field, $\sup_{z \in \partial\mathcal{D}} \|\phi(z)\|_{d+1} < \infty$ and $\inf_{z \in \partial\mathcal{D}} \langle \phi(z), n(z) \rangle > 0$.

Here $n(z)$ is the inwards pointing normal for $z \in \partial\mathcal{D}$.

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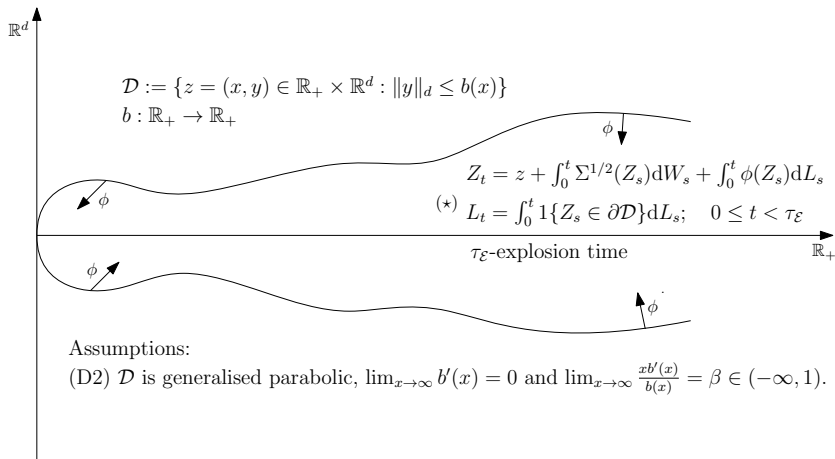
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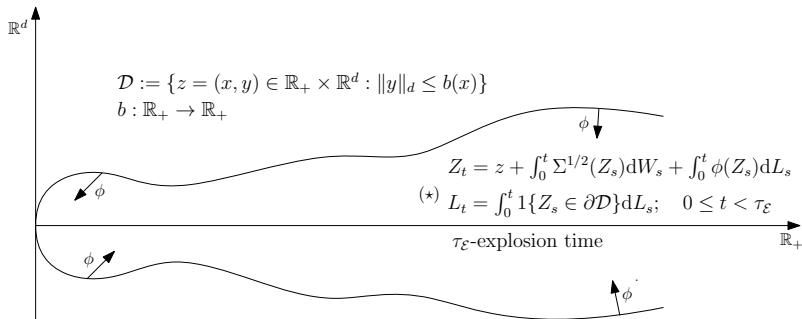
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[MMW22][Thm A.1]: existence and pathwise uniqueness hold for $(\star)$

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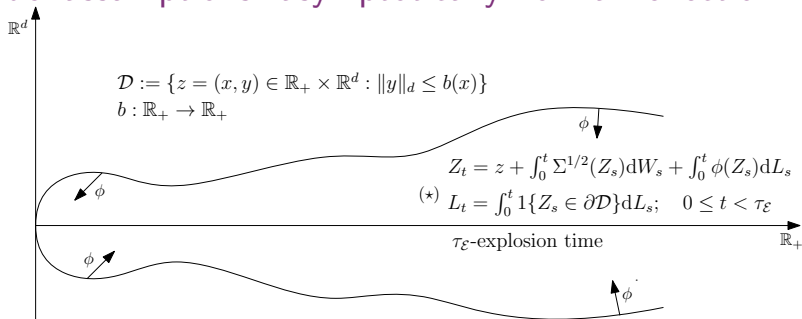


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(D2) $\mathcal{D}$ is generalised parabolic, $\lim_{x \rightarrow \infty} b'(x) = 0$ and $\lim_{x \rightarrow \infty} \frac{xb'(x)}{b(x)} = \beta \in (-\infty, 1)$.

(C2) $\lim_{x \rightarrow \infty} \langle e_x \Sigma(z), e_x \rangle = \sigma_1^2$ and $\lim_{x \rightarrow \infty} \text{Tr}(\Sigma(z)) - \sigma_1^2 = \sigma_2^2$, for some $\sigma_1^2, \sigma_2^2 \in (0, \infty)$.

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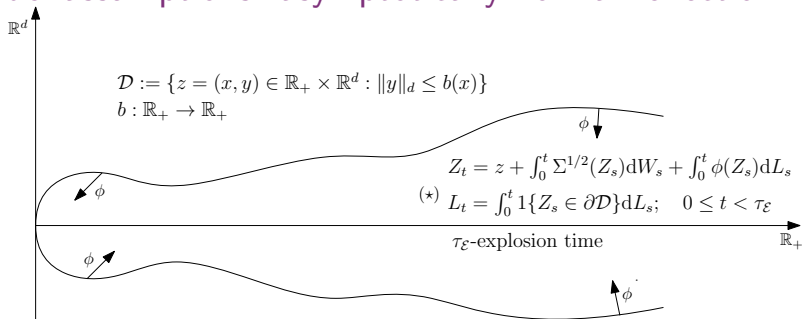
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Here $e_{\hat{y}} = (0, y/\|y\|_d) \in \mathbb{R}^d$ and normal $n(z) = \frac{b'(x)e_x - e_{\hat{y}}}{\sqrt{b'(x)^2 + 1}}$ for $z \in \partial \mathcal{D}$.

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- [MMW22][Thm A.1]: existence and pathwise uniqueness hold for (*)
- [BMW23][Thm 4.1]: $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty)$ for all $z \in \mathcal{D}$ and Z is diffusive:

$$\sup_{t \in \mathbb{R}_+} \mathbb{E}_z \|Z_t\|_{d+1}^2 / (1+t) < \infty.$$

Phase transition: recurrence/transience

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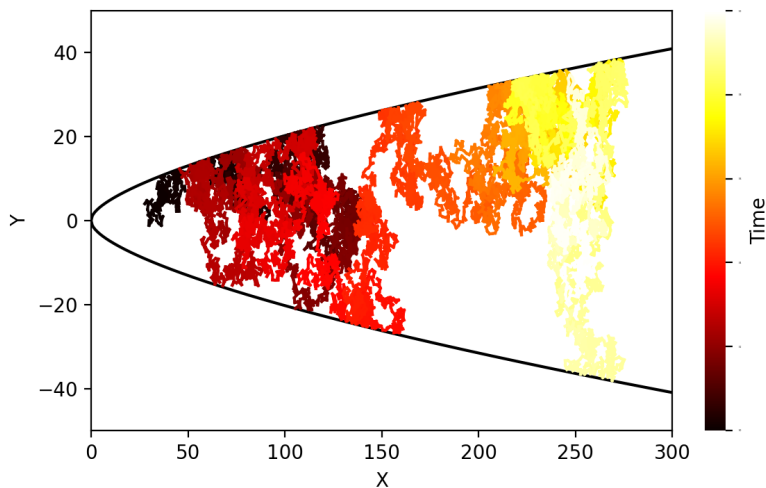
Theorem ([BMW23])

Suppose that Assumptions (D2), (C2) and (V2) hold and the process Z solves SDE $(\star)$. Then the following statements hold for all starting points $z \in \mathcal{D}$:

- (a) *if $\beta \in (-\infty, \beta_c)$, then Z is recurrent;*
- (b) *if $\beta \in (\beta_c, 1)$, then Z is transient;*
- (c) *if $\beta = \beta_c$ and additional assumptions are satisfied, then Z is recurrent.*

This theorem extends [Pin09] where $(d+1)$ -dimensional BM with normal reflection was studied for $\beta \geq 0$. In that case $\beta_c = 1/d$.

Transience: $\beta = 0.65 > 0.5 = \beta_c$



Martingale in interior, pushed away at boundary. Normal reflection, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 1$. (If $\sigma_1^2 = \sigma_2^2 = 1$ in $\mathcal{D} \subset \mathbb{R}^2$, Z not transient.)

Return times of the reflected process

Tails of return times of $Z = (X, Y) \in \mathbb{R}_+ \times \mathbb{R}^d$ below any level $r > 0$,

$$\varsigma_r := \inf\{t \in \mathbb{R}_+ : X_t \leq r\}.$$

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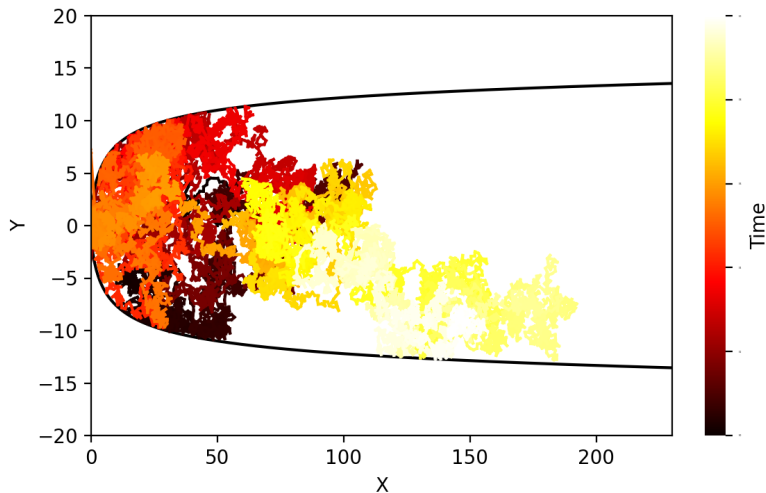
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- (b) $\beta \in (-\infty, \beta_c)$: *for any $\varepsilon > 0$, level $r > 0$ and starting point $z \in \mathcal{D} \cap (r, \infty) \times \mathbb{R}^d$, there exist $C_1, C_2 \in (0, \infty)$ such that*

$$C_1 t^{-m_c - \varepsilon} \leq \mathbb{P}_z(\varsigma_r \geq t) \leq C_2 t^{-m_c + \varepsilon} \quad \text{for all } t \in (1, \infty).$$

Null recurrence: $\beta = 0.1 < 1 = \beta_c$ ($m_c = 0.45$)



Normally reflected BM in an unbounded expanding domain. Tail of return time decays as $t^{-0.45}$, $t \rightarrow \infty$. ("less" recurrent than scalar BM.)

Invariant distribution and the rate of convergence

Theorem ([BMW23])

Let (D2), (C2), (V2) hold & $\beta < -\beta_c$. Then reflected process Z in $(\star)$, has a unique invariant distribution π . Define

$M_c := -(1 + \beta/\beta_c)/2 > 0$ and pick $\epsilon > 0$.

(a) $\exists$ constants $c_\pi, C_\pi \in (0, \infty)$ such that, for $r \in [1, \infty)$,

$$c_\pi r^{-2M_c - \epsilon} \leq \pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \leq C_\pi r^{-2M_c + \epsilon};$$

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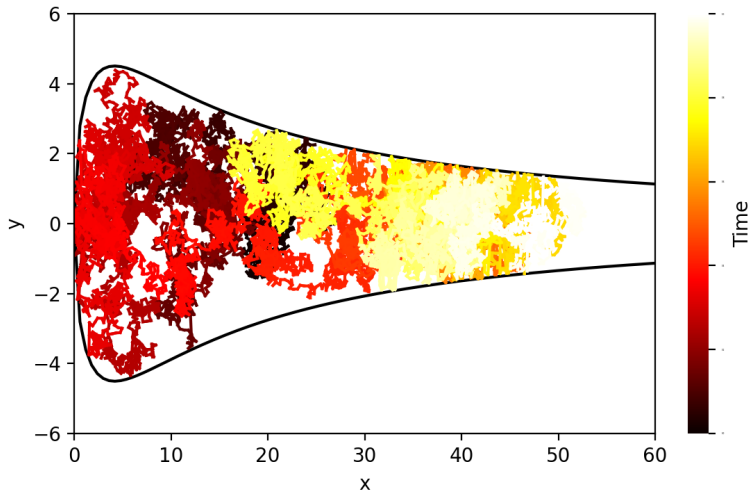
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$$\lim_{t \rightarrow \infty} \frac{\log \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{TV}}{\log t} = \frac{1}{2} \lim_{r \rightarrow \infty} \frac{\log \pi(\{z \in \mathcal{D} : \|z\|_{d+1} > r\})}{\log r} = -M_c.$$

Positive recurrence: $\beta = -1.2 < -1 = -\beta_c$ ($M_c = 0.1$)



Normally reflected Brownian motion in an unbounded narrowing domain.
Tail decay: $\pi(\|z\|_2 > a) \approx a^{-0.2}$; Convergence in TV: $t^{-0.1}$ as $t \rightarrow \infty$.

Remarks

Heuristic dynamics:

$$d\tilde{X}_t = \frac{\beta}{\beta_c \tilde{X}_t} dt + d\tilde{W}_t, \quad \text{for large times } t \text{ and values } \tilde{X}_t$$

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Novelty of our approach:

- 1 Normally reflected BM exhibits behaviours from **transience to stability** in **any dimension** with **no interior drift**.
- 2 **Single class** of Lyapunov functions covers all non-critical cases.
- 3 **Lower bounds** on the **tail** of invariant distribution and on the **rate** of convergence, that **match upper bounds**.

Proofs

- Intuition: see Part II of the presentation on this channel!

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- Details: M. BREŠAR, A. MIJATOVIĆ, & A.R. WADE, [Brownian motion with asymptotically normal reflection in unbounded domains: from transience to stability](#), see link to arXiv below the video

Thank you for your attention!!