

Brownian motion with asymptotically normal reflection in unbounded domains: from transience to stability

(Part II: intuition behind the proofs)

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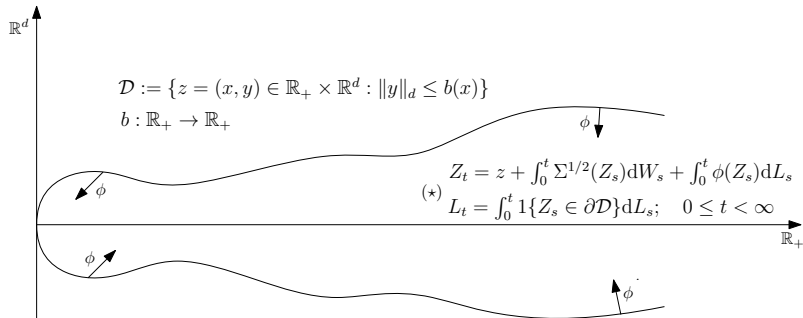
Joint work with Miha Brešar and Andrew Wade

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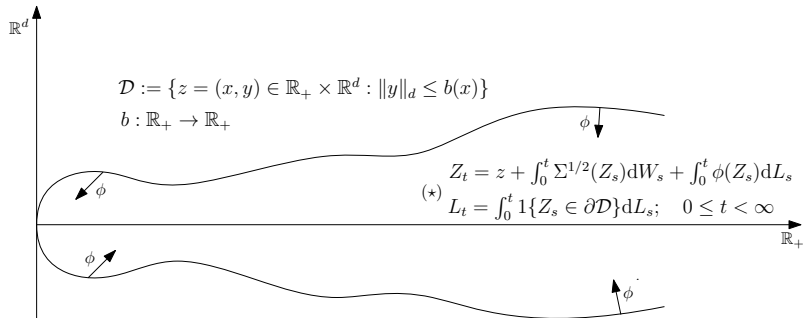
Model assumptions: asymptotically normal reflection



Assumptions:

- (D2) $\mathcal{D}$ is generalised parabolic, $\lim_{x \rightarrow \infty} b'(x) = 0$ and $\lim_{x \rightarrow \infty} \frac{xb'(x)}{b(x)} = \beta \in (-\infty, 1)$.
- (C2) $\lim_{x \rightarrow \infty} \langle e_x \Sigma(z), e_x \rangle = \sigma_1^2$ and $\lim_{x \rightarrow \infty} \text{Tr}(\Sigma(z)) - \sigma_1^2 = \sigma_2^2$, for some $\sigma_1^2, \sigma_2^2 \in (0, \infty)$.
- (V2) $\langle \phi(z), e_x \rangle = b'(x)s_0 + o(1)$ and $\langle \phi(z), -e_{\hat{y}} \rangle = c_0 + o(1)$, $x \rightarrow \infty$, for $s_0, c_0 \in (0, \infty)$.

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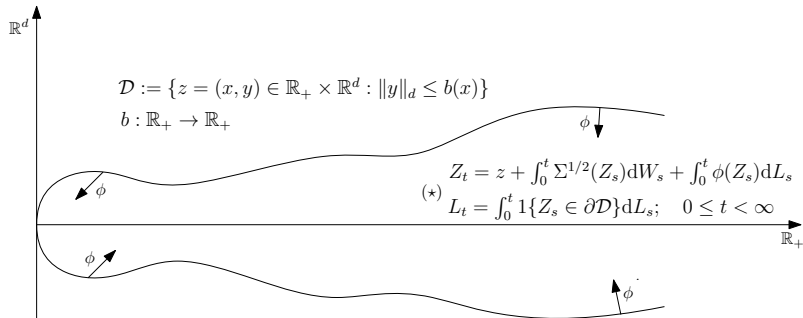


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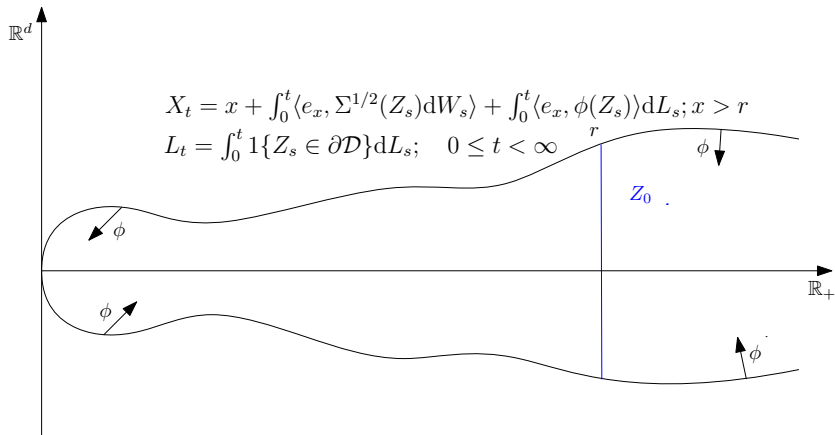
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- [MMW22][Thm A.1]: pathwise unique solution exists
- [BMW23][Thm 4.1]: Z is diffusive and $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty)$ for all $z \in \mathcal{D}$.

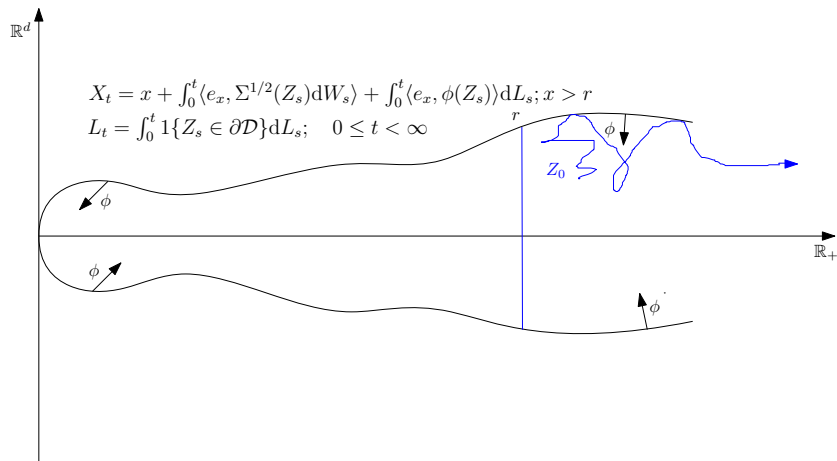
Structure of proofs

Our approach: deduce the long-term behaviour of Z from return times to the compact sets!



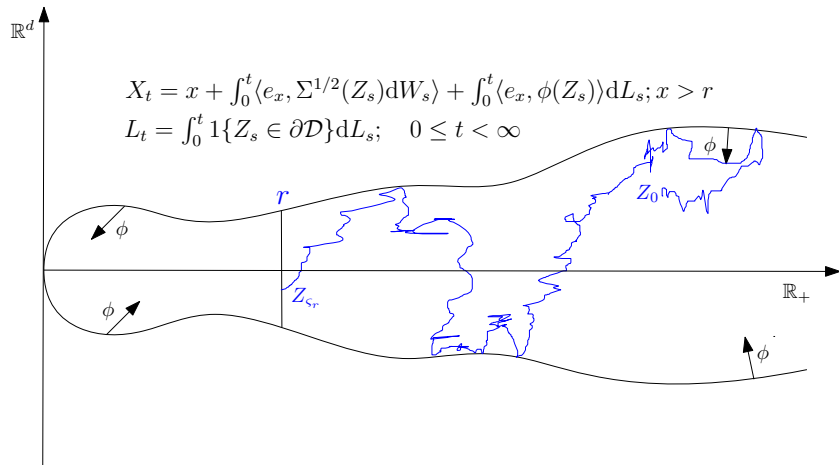
Structure of proofs - transience $\beta \in (\beta_c, 1)$

Recall $\beta_c = c_0\sigma_1^2/(s_0\sigma_2^2)$. We prove for every $Z_0 \in (r, \infty) \times \mathbb{R}^d$, the return time $\mathbb{P}_z(\zeta_r = \infty) > 0$



Structure of proofs - recurrence $\beta \in (-\infty, \beta_c)$

Recall $m_c := \frac{1}{2}(1 - \beta/\beta_c)$. We classify the tails of return time
 $ct^{m_c - \varepsilon} \leq \mathbb{P}_z(\varsigma_{r_0} > t) \leq Ct^{m_c + \varepsilon}$.



$$X_t = x + \int_0^t \langle e_x, \Sigma^{1/2}(Z_s) dW_s \rangle + \int_0^t \langle e_x, \phi(Z_s) \rangle dL_s; x > r$$

$$L_t = \int_0^t 1\{Z_s \in \partial\mathcal{D}\} dL_s; \quad 0 \leq t < \infty$$

Structure of proofs - positive recurrence $\beta \in (-\infty, -\beta_c)$

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What super/submartingale are suitable? **Inhomogeneity** of state space $\mathcal{D}$

A Lyapunov function

For any $\gamma \in \mathbb{R}$ and $w \in \mathbb{R} \setminus \{0\}$, let

$$f_{w,\gamma}(x,y) := (x + k_w)^\gamma \left(1 + w(1-w) \frac{\|y\|_d^2}{2(x + k_w)^2} \right)^{\gamma/w}.$$

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Consider the process $\kappa_t := f_{w,\gamma}(Z_t)$. Itô's formula:

$$df_{w,\gamma}(Z_t) = \frac{1}{2} \Delta_\Sigma f_{w,\gamma}(Z_t) dt + \langle \nabla f_{w,\gamma}(Z_t), \phi(Z_t) \rangle dL_t + dM_t, \text{ for } 0 \leq t < \infty,$$

where

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- define $w_0 := 1 - s_0\beta/c_0$, then for large $x \in \mathbb{R}_+$ we have

$$\begin{cases} \langle \nabla f_{w,\gamma}, \phi(x, y) \rangle \leq 0, & \text{if } w > w_0, \\ \langle \nabla f_{w,\gamma}, \phi(x, y) \rangle \geq 0, & \text{if } w < w_0. \end{cases}$$

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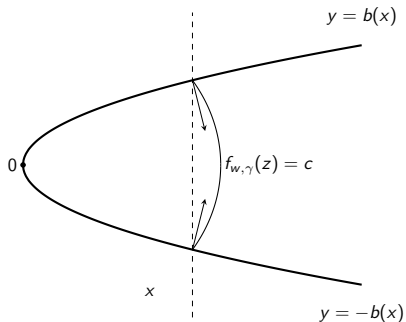
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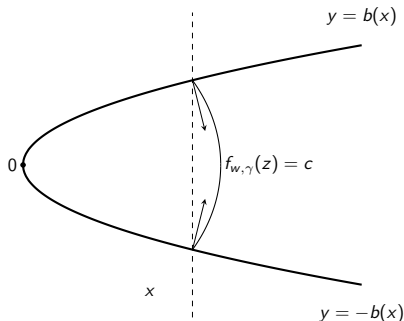
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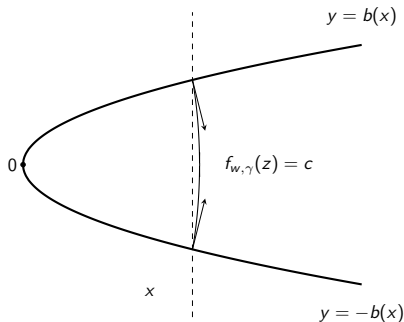
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Intuition: For $w < w_0$, the level set of $f_{w,\gamma}$ is such that $\langle \nabla f_{w,\gamma}(z), \phi(z) \rangle < 0$ for all $z = (x, y)$ with large x .



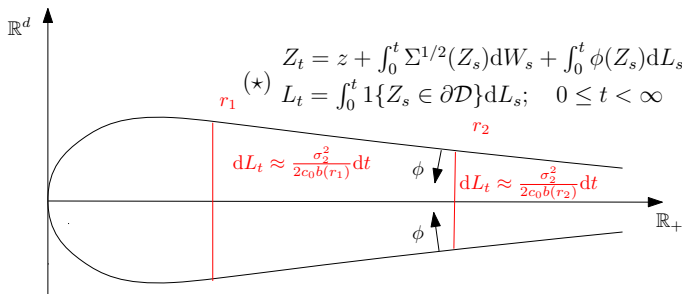
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Technical difficulties

The super/submartingale criteria provide precise characterisation of **tails of return times**. But **ergodicity** requires also local behaviour, e.g. **Feller continuity** and **irreducibility**.



Feller continuity is a technical challenge. In a **shrinking** domain, **local time** accumulates **very fast**! Classical approach of [SV71] does not work.

Closing remarks

- This paper extends results from [Pin09], who studied the case with normal reflection, $\Sigma \equiv \text{Id}$ and non-shrinking domains ($\beta \geq 0$), observing transitions between transience and recurrence only.

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- Super/submartingale methods for lower bounds for tails of return times developed for the reflected diffusions are applicable to general continuous semimartingales and thus may yield lower bounds on tails of invariant distributions in other contexts.

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