

Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity

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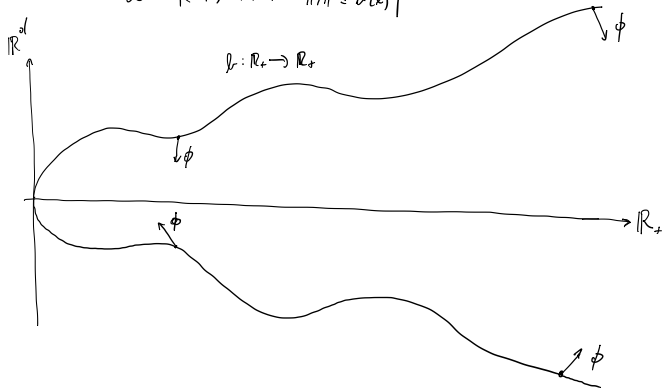
Joint work with
Mikhail Menshikov and Andrew Wade

EPSRC

Engineering and Physical Sciences
Research Council

EP/V009478/1 & EP/P003818/1

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 $\mathbb{R}^d$

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$$L_t = \int_0^t \mathbb{I}_{\{z_s \in \partial \mathcal{D}\}} dL_s; \quad t \in [0, \tau_{\mathcal{E}})$$

$\tau_{\mathcal{E}} = \text{EXPLOSION TIME}$

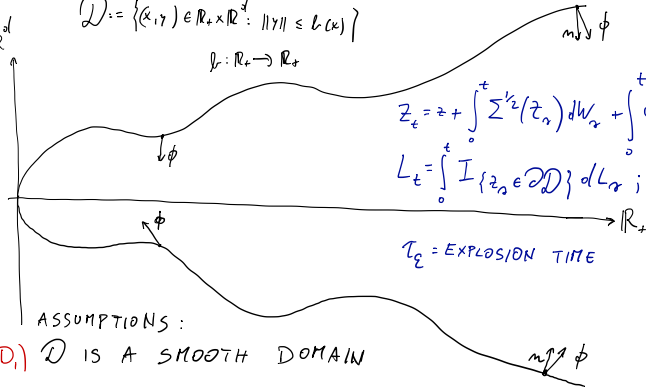
(W STANDARD BM IN $\mathbb{R}^{d+1}$)

EXISTENCE AND UNIQUENESS

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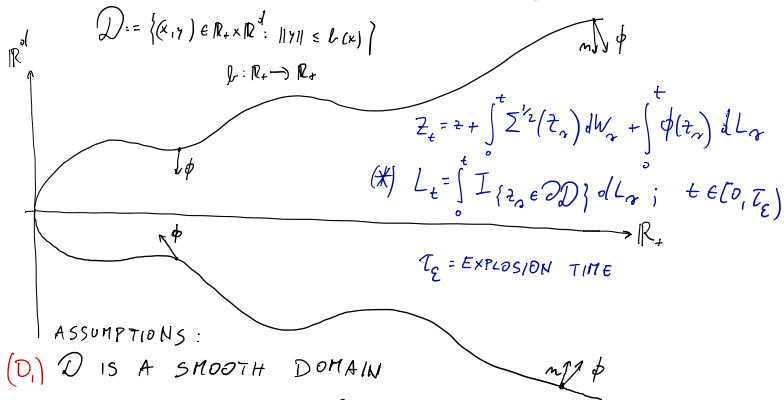
$\nwarrow \phi$

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(V) $\phi: \partial D \rightarrow \mathbb{R}^{d+1}$ C^2 -VECTOR FIELD: $\sup_{z \in \partial D} \|\phi(z)\| < \infty$ & $\inf_{z \in \partial D} \langle \phi(z), m(z) \rangle > 0$
 ($m(z) :=$ INWARD NORMAL AT $z \in \partial D$)

Existence and uniqueness

Theorem (MMW 2022)

Suppose that (D1), (C), and (V) hold. Then there exists a strong solution $(Z, L, \tau_{\mathcal{E}})$ satisfying (), and there is pathwise uniqueness. In particular, (*) defines a continuous strong Markov process Z over time interval $[0, \tau_{\mathcal{E}})$ and*

$$\lim_{t \uparrow \tau_{\mathcal{E}}} \|Z_t\| = \lim_{t \uparrow \tau_{\mathcal{E}}} L_t = \infty, \text{ on } \{\tau_{\mathcal{E}} < \infty\}.$$

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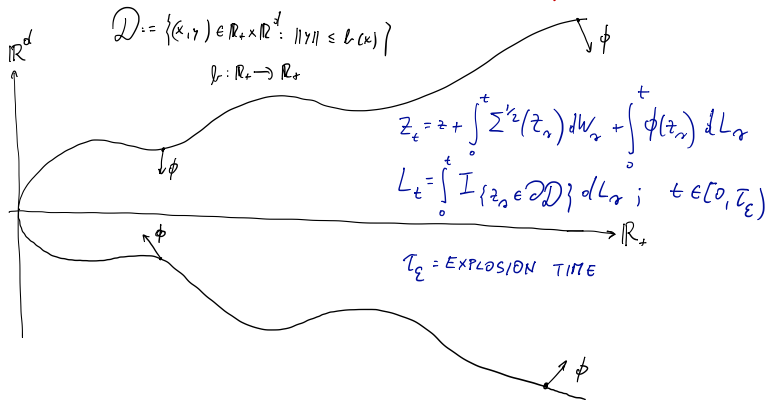
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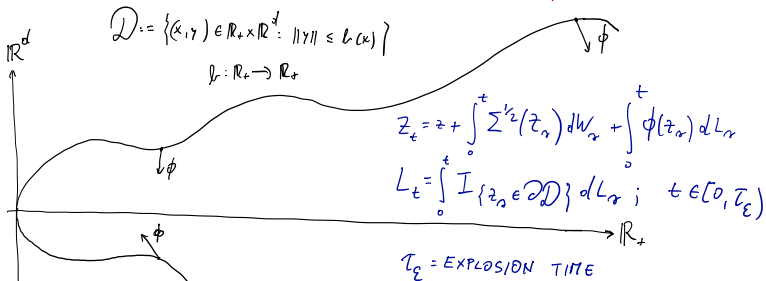
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- Proof via approximation by increasing sequence of bounded domains and applying existence/uniqueness results of LIONS & SZNITMAN, 1984. Main technical obstacle is verifying $\lim_{t \uparrow \tau_{\mathcal{E}}} \|Z_t\| = \infty$ on $\{\tau_{\mathcal{E}} < \infty\}$, i.e., explosive paths do not oscillate.

EXPLOSION AND SUPERDIFFUSIVITY



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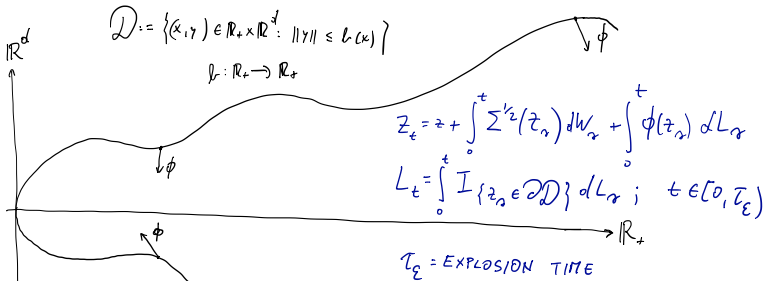


ASSUMPTIONS

(D₂) D IS GENERALISED PARABOLIC:

$$b'(x) \xrightarrow{x \rightarrow \infty} 0 \quad \& \quad \beta := \limsup_{x \rightarrow \infty} \frac{x b'(x)}{b(x)} \in (-\infty, 1) \quad \& \quad \lim_{x \rightarrow \infty} b''(x) = \lim_{x \rightarrow \infty} b''(x) b(x) = 0$$

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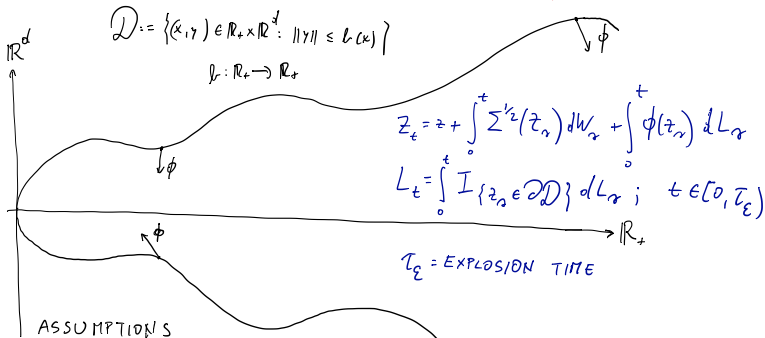
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$\exists s_0, c_0 \in (0, \infty)$ SUCH THAT $\langle m^\perp(z), \phi(z) \rangle \xrightarrow{x \rightarrow \infty} s_0$ & $\langle m(z), \phi(z) \rangle \xrightarrow{x \rightarrow \infty} c_0$

Explosion and strong law

Last piece of notation: for $x \in \mathbb{R}_+$,

$$B(x) := \int_0^x b(u)du, \text{ and } B(\infty) = \lim_{x \rightarrow \infty} B(x) \in [0, \infty].$$

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Explosion: If $B(\infty) < \infty$, then $\sup_{z \in \mathcal{D}} \mathbb{E}_z \tau_{\mathcal{E}} < \infty$, and

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Non-explosion: If $B(\infty) = \infty$, then, for every $z \in \mathcal{D}$, $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$, and we have the **strong law**

$$\lim_{t \rightarrow \infty} \frac{B(X_t)}{t} = \lim_{t \rightarrow \infty} \frac{B(\|Z_t\|_{d+1})}{t} = \frac{s_0 \sigma^2}{2c_0}, \mathbb{P}_z\text{-a.s.}$$

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Theorem 2 then yields the following.

- If $\beta < -1$, then $\mathbb{E}_z \tau_{\mathcal{E}} < \infty$, i.e., **explosion** occurs.

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- If $\beta \in (-1, 1)$, then $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$, and, $\mathbb{P}_z$ -a.s.,

$$\lim_{t \rightarrow \infty} t^{-\frac{1}{1+\beta}} X_t = \left(\frac{(1+\beta)s_0 \sigma^2}{2ac_0} \right)^{\frac{1}{1+\beta}}. \quad (\text{superdiffusive transience})$$

Explosion and strong law - Remarks

- For any $\beta < 1$, we have $X_t/\sqrt{t} \rightarrow \infty$, a.s.; this **superdiffusive transience** is an example of the phenomenon of **anomalous diffusion**.

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- **Existence** results for diffusions with oblique reflections (LIONS & SZNITMAN, 1984; STROOCK & VARADHAN, 1971) do not admit explosion (not applicable to unbounded, narrowing domain).

Proofs

- Intuition: see Part II of this presentation on this channel!

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- Details: M. MENSNIKOV, A. MIJATOVIĆ, & A.R. WADE, Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity. *Ann. Inst. Henri Poincaré Probab. Stat.* (2022), ArXiv: 2203.00966.

Thank you for your attention!!