

Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity (Part II: intuition behind the proofs)

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Joint work with
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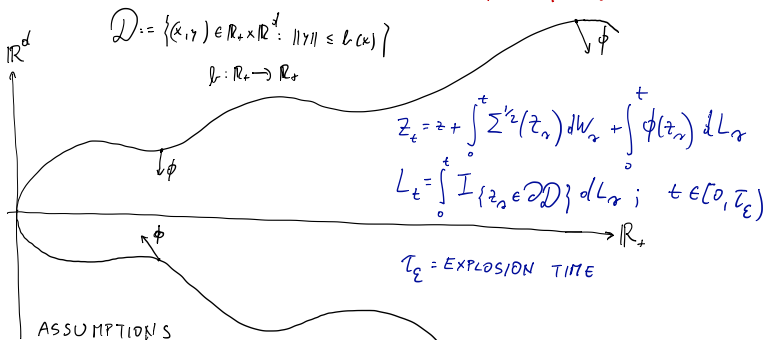
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Main aim

M. MENSNIKOV, A. MIJATOVIĆ, & A.R. WADE, Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity. *Ann. Inst. Henri Poincaré Probab. Stat.* (2022), ArXiv: 2203.00966.

EXPLOSION AND SUPERDIFFUSIVITY



(D₂) $\mathcal{D}$ IS GENERALISED PARABOLIC:

$$b'(x) \xrightarrow{x \rightarrow \infty} 0 \quad \text{AND} \quad \beta := \limsup_{x \rightarrow \infty} \frac{x b'(x)}{b(x)} \in (-\infty, 1)$$

(A) $\exists \sigma^2 \in (0, \infty)$ SUCH THAT $T_L(\Sigma(z)) - \Sigma_{11}(z) \xrightarrow{x \rightarrow \infty} \sigma^2 \quad (z = (x, y) \in \mathcal{D}) \subseteq \mathbb{R}_+ \times \mathbb{R}^d$

$\exists s_0, c_0 \in (0, \infty)$ SUCH THAT $\langle m^\perp(z), \phi(z) \rangle \xrightarrow{x \rightarrow \infty} s_0$ & $\langle m(z), \phi(z) \rangle \xrightarrow{x \rightarrow \infty} c_0$

Explosion and strong law

Last piece of notation: for $x \in \mathbb{R}_+$,

$$B(x) := \int_0^x b(u)du, \text{ and } B(\infty) = \lim_{x \rightarrow \infty} B(x) \in [0, \infty].$$

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Theorem (MMW 2022)

Suppose that (D1), (D2), (C), (V), and (A) hold.

Explosion: If $B(\infty) < \infty$, then $\sup_{z \in \mathcal{D}} \mathbb{E}_z \tau_{\mathcal{E}} < \infty$, and

$$\lim_{t \uparrow \tau_{\mathcal{E}}} X_t = \lim_{t \uparrow \tau_{\mathcal{E}}} L_t = \infty, \mathbb{P}_z\text{-a.s. for every } z \in \mathcal{D}.$$

Non-explosion: If $B(\infty) = \infty$, then, for every $z \in \mathcal{D}$, $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$, and we have the **strong law**

$$\lim_{t \rightarrow \infty} \frac{B(X_t)}{t} = \lim_{t \rightarrow \infty} \frac{B(\|Z_t\|_{d+1})}{t} = \frac{s_0 \sigma^2}{2c_0}, \mathbb{P}_z\text{-a.s.}$$

A Lyapunov function

It's not clear how to turn the preceding heuristic into a proof. Instead, we use a **Lyapunov function** $g : \mathcal{D} \rightarrow \mathbb{R}_+$ to project and homogenize.

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Define $g(z) := g(x, y)$ for $z = (x, y) \in \mathbb{R}_+ \times \mathbb{R}^d$ by

$$g(x, y) := x + \gamma \frac{\|y\|_d^2}{b(x)}, \text{ for a parameter } \gamma \in \mathbb{R}.$$

(Technicality: Need some smoothing near $x = 0$; ignore this here.)

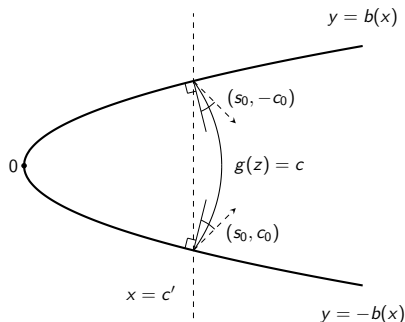
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Intuition: $g(x, y) \approx x$ but, while X has zero drift in the interior, the curved level sets of g produce (positive but small) drift for $g(Z)$, while γ can be tuned to control the sign of the local-time drift arising from the reflection, as it modulates the curvature of the level set $\{z \in \mathcal{D} : g(z) = c\}$. Sign depends on inner product of ϕ and level lines of g .

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Recall $g(x, y) := x + \gamma \frac{\|y\|_d^2}{b(x)}$, for a parameter $\gamma \in \mathbb{R}$.

The key value of γ will turn out to be $\gamma_0 := s_0/(2c_0)$.

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Consider the process $\xi_t := B(g(Z_t))$. Itô's formula:

$$dB(g(Z_t)) = \mu(Z_t)dt + b(g(Z_t))\nu(Z_t)dL_t + dM_t, \text{ for } 0 \leq t < \tau_{\mathcal{E}},$$

where

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Roughly this means that

- for $\gamma > \gamma_0$, $\xi_t - (\gamma\sigma^2 + \varepsilon)t$ is a **supermartingale**;
- for $\gamma \in (0, \gamma_0)$, $\xi_t - (\gamma\sigma^2 - \varepsilon)t$ is a **submartingale**.

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- Control of coefficients in the SDE is as $x \rightarrow \infty$, so **stop on the left**.
- To get **true martingale** from **local martingale**, **stop on the right**.
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Ignoring technicalities, let $\sigma_r := \inf\{t \in \mathbb{R}_+ : X_t \geq r\}$ (since $g(Z_t) \approx X_t$ we can think of this as passage time for $g(Z)$ instead). Then, by **optional stopping** and the ξ -sup/supermartingale,

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Note $\tau_{\mathcal{E}} = \lim_{r \rightarrow \infty} \sigma_r$, so $B(\infty)$ relates to $\mathbb{E} \tau_{\mathcal{E}}$.

Passage-time asymptotics

A key step in following through this scheme is combine the facts that (i) $\mathbb{E}_z \sigma_r < \infty$, i.e., we **exit bounded sets** in reasonable time; and (ii) starting from away from a bounded set, have uniformly positive probability to never return, i.e., **transience**.

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- To prove (i), we use our conditions that **reflection** eventually takes one away from the boundary, and then **ellipticity** gives positive probability of exiting the bounded domain in a reasonable time.

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Combining all these elements:

Proposition

Suppose that (D1), (D2), (C), (V), and (A) hold.

If $B(\infty) < \infty$, then $\sup_{z \in \mathcal{D}} \mathbb{E}_z \tau_{\mathcal{E}} < \infty$.

On the other hand, if $B(\infty) = \infty$, then, for all $z \in \mathcal{D}$,

$$\lim_{r \rightarrow \infty} \frac{\mathbb{E}_z \sigma_r}{B(r)} = \frac{2c_0}{s_0 \sigma^2}.$$

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One advantage of this 'sandwich' method, using a submartingale for one direction and a supermartingale for the other, is that we only ever need to control the **sign** of the local time drift, so we do not need to do any (potentially hard) quantitative analysis of the local time. Indeed, a by-product of our strong law is a result on **growth asymptotics** of local time (in the non-explosive case).

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In the paper we formulate these ideas in a fairly general framework in terms of **semimartingale criteria** for potentially explosive, right-continuous, adapted processes on $\mathbb{R}_+$.

Closing remarks

- The term **anomalous diffusion** has been in use in the physics literature since at least the 1980s; it gained its own AMS MSC code (60K50) in the 2020 revision.

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- For general Σ , one gets richer behaviour, in that transience and recurrence, even **positive recurrence** in the case of a shrinking domain, can be achieved by changing Σ even while keeping all other parameters fixed. Moreover, only **asymptotic normality** matters. A joint paper with M. BREŠAR and A. WADE on this topic will be released soon.

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Thank you for wathcing!