

Hölder continuity of the convex minorant of a Lévy process

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Path of a Lévy process X and its convex minorant C

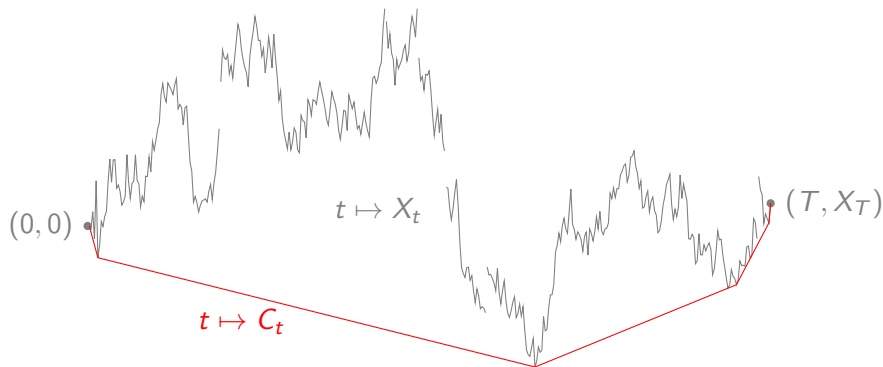


Figure: In [1] we characterise, in terms of transition probabilities of X , when $t \mapsto C_t$ is smooth and prove that $t \mapsto C_t$ is Lipschitz if and only if X is of finite variation.

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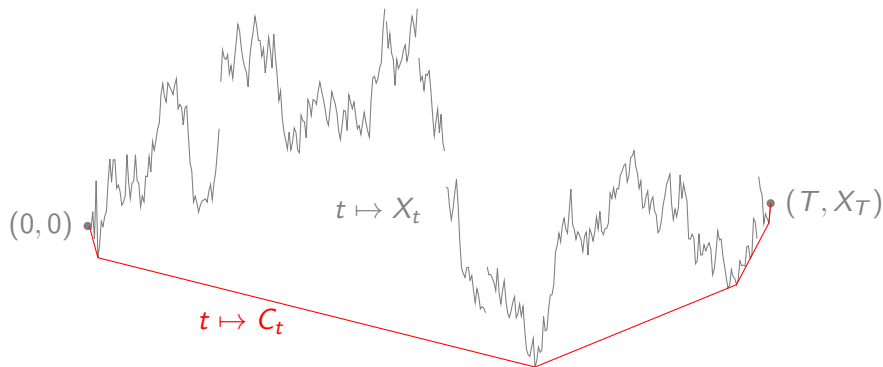


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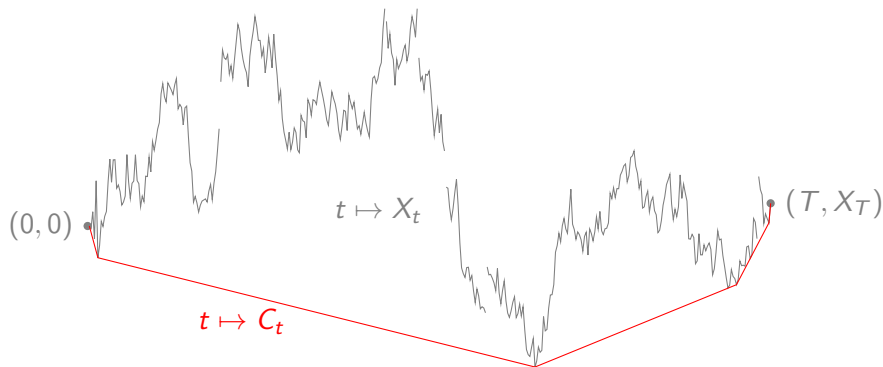


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Put differently, is the Hölder constant $\sup_{0 \leq u < t \leq T} |C_t - C_u| / (t - u)^r$ finite a.s.?

Main results: when is C r -Hölder continuous?

Let ν be Lévy measure of X , $\bar{\nu}(x) := \nu(\mathbb{R} \setminus (-x, x))$ and Blumenthal–Gettoor index

$$\beta := \inf \left\{ p > 0 : \int_0^1 \bar{\nu}(x^{1/p}) dx < \infty \right\}. \quad \text{Note } \int_0^1 \bar{\nu}(x^{1/\beta}) dx \text{ need not be finite!}$$

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Lévy process X			$r \in (0, 1)$	Is C r -Hölder continuous?
$\sigma^2 > 0$ (Brownian component present)			$0 < r < 1/2$	Yes
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Table: Critical Hölder exponent: $1/2 \cdot \mathbb{1}_{\sigma^2 > 0}$

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$$\int_0^1 \bar{\nu}(x^{1/\beta}) dx < \infty \ \& \ \int_0^1 x \bar{\nu}(x^{1/\beta})^2 dx < \infty \implies I_\beta < \infty.$$

Controlling the Hölder constant

- * The convex minorant C is piecewise linear with countably many maximal intervals of linearity with horizontal lengths $(\ell_n)_{n \in \mathbb{N}}$ and vertical heights $(\xi_n)_{n \in \mathbb{N}}$, see e.g. [3].

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- * Key inequalities for the Hölder constant:

$$k_r := \sup_{s \in \mathcal{S}_r} |s| \leq \sup_{0 \leq u < t \leq T} \frac{|C_t - C_u|}{(t - u)^r} \leq \left(\sum_{s \in \mathcal{S}_r} |s|^{1/(1-r)} \right)^{1-r} =: K_r \quad \text{a.s.}$$

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- * Our main results show that, for most Lévy processes, k_r and K_r are simultaneously finite or infinite, i.e., $\mathbb{P}(\{K_r = \infty\} \cap \{k_r < \infty\}) = 0$, yielding Table 1.

Structure of proof: away from criticality ($r \in (0, 1) \setminus \{1/\beta\}$ and $\sigma^2 = 0$)

Proposition 1 (0–1 law for unbounded functionals of the faces of C)

Let $\phi : \mathbb{R} \times (0, \infty) \rightarrow [0, \infty)$ be measurable, then the sum $\sum_{n \in \mathbb{N}} \phi(\xi_n, \ell_n)$ is either a.s. finite or a.s. infinite. Moreover, we have

$$\sum_{n \in \mathbb{N}} \phi(\xi_n, \ell_n) < \infty \quad \text{a.s.} \quad \Longleftrightarrow \quad \int_0^1 \mathbb{E}[\min\{\phi(X_t, t), 1\}] \frac{dt}{t} < \infty.$$

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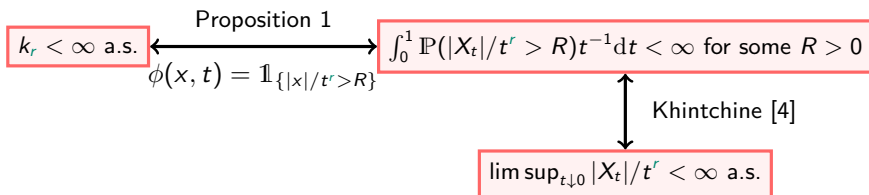
$$\boxed{k_r < \infty \text{ a.s.}} \xleftrightarrow[\phi(x, t) = \mathbb{1}_{\{|x|/t^r > R\}}]{\text{Proposition 1}} \boxed{\int_0^1 \mathbb{P}(|X_t|/t^r > R) t^{-1} dt < \infty \text{ for some } R > 0}$$

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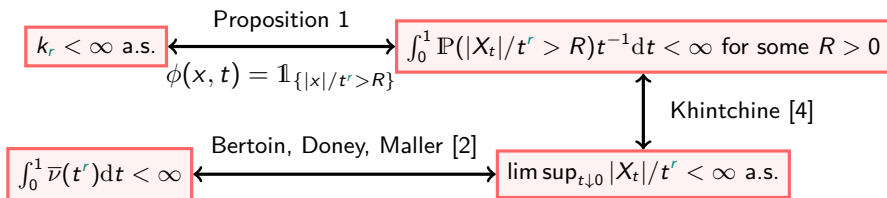


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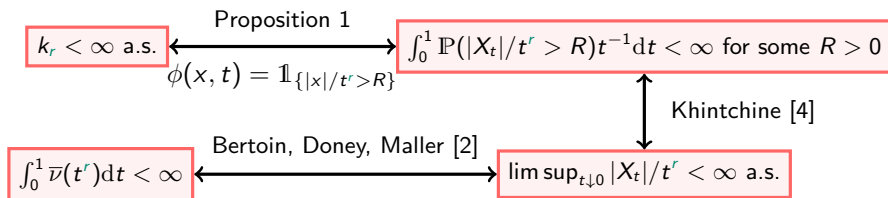


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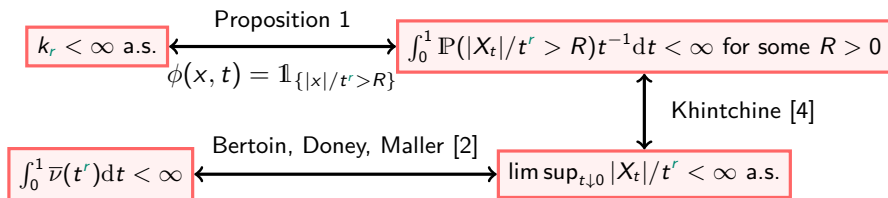
For $r' > r$, we have $K_r^{1/(1-r)} = \sum_{n \in \mathbb{N}} (|\xi_n|/\ell_n^r)^{1/(1-r)} \leq k_{r'}^{1/(1-r)} \sum_{n \in \mathbb{N}} \ell_n^{(r'-r)/(1-r)}.$

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$$k_{r'} < \infty \implies K_r < \infty$$

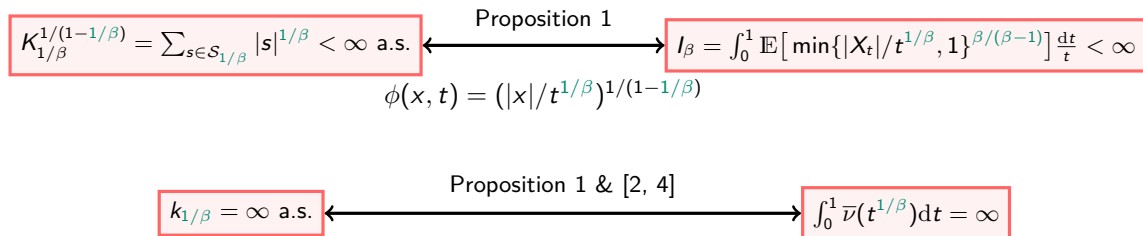
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In the critical case, we show that $\int_0^1 \bar{\nu}(t^{1/\beta}) dt = \infty \iff k_{1/\beta} = \infty$ a.s. and $l_\beta < \infty \iff K_{1/\beta} < \infty$ a.s.:

$$\boxed{K_{1/\beta}^{1/(1-1/\beta)} = \sum_{s \in S_{1/\beta}} |s|^{1/\beta} < \infty \text{ a.s.}} \xleftrightarrow{\text{Proposition 1}} \boxed{l_\beta = \int_0^1 \mathbb{E}[\min\{|X_t|/t^{1/\beta}, 1\}^{\beta/(\beta-1)}] \frac{dt}{t} < \infty}$$
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- [1] David Bang, Jorge González Cázares, and Aleksandar Mijatović. “When is the convex minorant of a Lévy process smooth?” (2022). [arXiv: 2205.14416v2 \[math.PR\]](#).
- [2] J. Bertoin, R. A. Doney, and R. A. Maller. “Passage of Lévy processes across power law boundaries at small times”. *Ann. Probab.* 36.1 (2008), pp. 160–197. ISSN: 0091-1798. DOI: [10.1214/009117907000000097](#). URL: <https://doi.org/10.1214/009117907000000097>.
- [3] Jorge Ignacio González Cázares and Aleksandar Mijatović. “Convex minorants and the fluctuation theory of Lévy processes”. *ALEA, Lat. Am. J. Probab. Math. Stat.* 19 (2022), pp. 983–999. DOI: [10.30757/ALEA.v19-39](#).
- [4] A. Khintchine. “Sur la croissance locale des processus stochastiques homogènes à accroissements indépendants”. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* 1939 (1939), pp. 487–508.

Link to arXiv in the description of the video.

Thank you for your attention!!