

How smooth can the convex hull of a Lévy path be?

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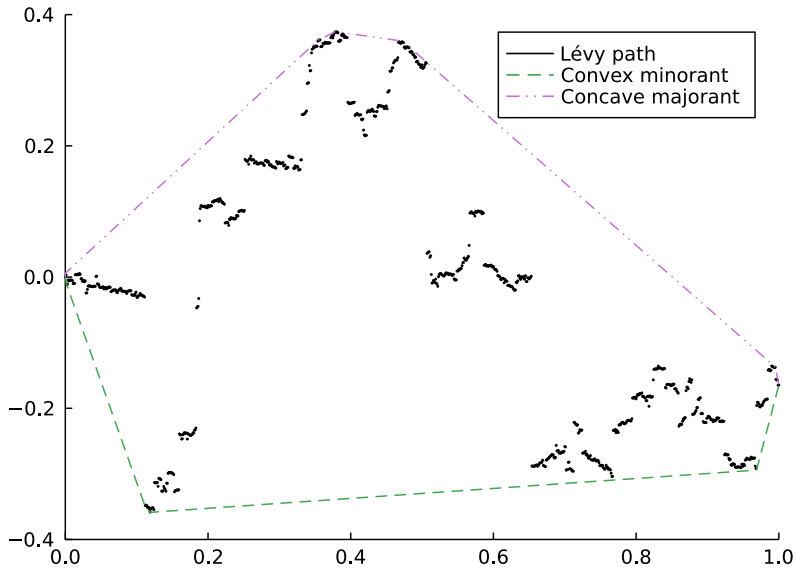


Figure: Smooth convex hull of a Lévy path [2] and [Youtube video](#)

Convex minorant C , its derivative C' and vertex time τ_s (finite variation)

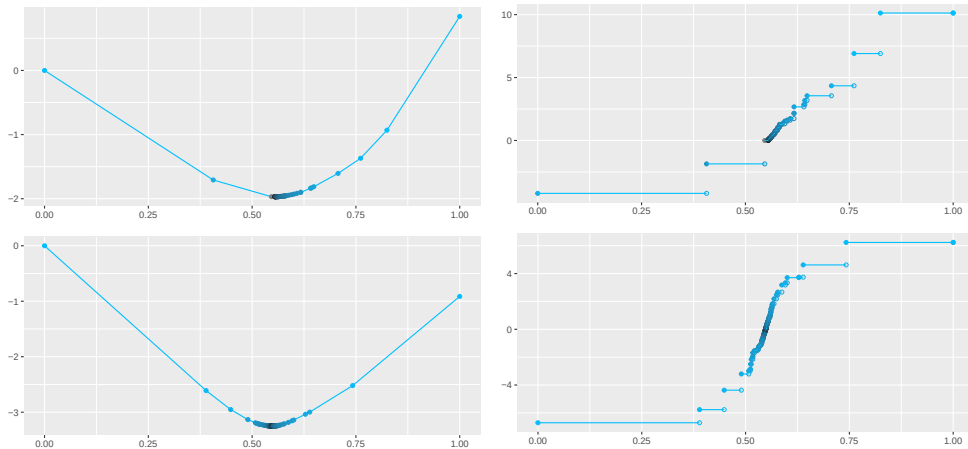


Figure: Convex minorant C of a finite variation Lévy process is pictured on the left, with right-derivative C' on the right. In the both cases $0 \in \mathcal{L}^+(\mathcal{S})$ and C' is not constant immediately after τ_0 .

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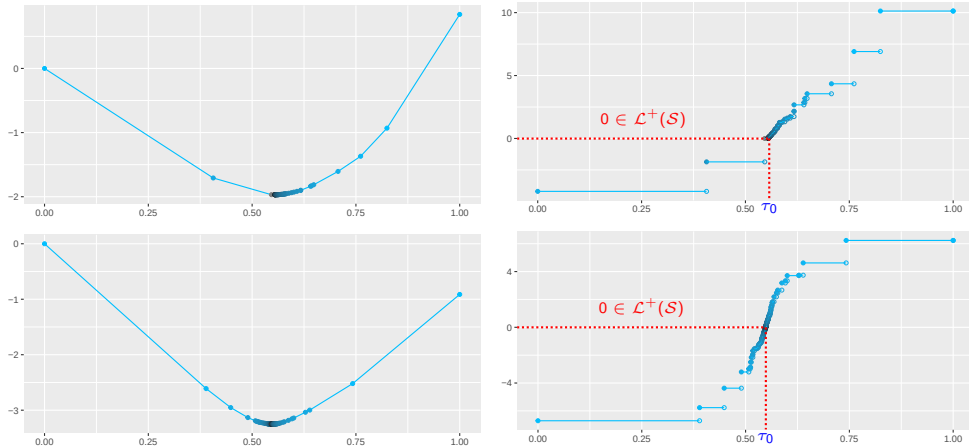


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Convex minorant C , its derivative C' and vertex time τ_s (infinite variation)

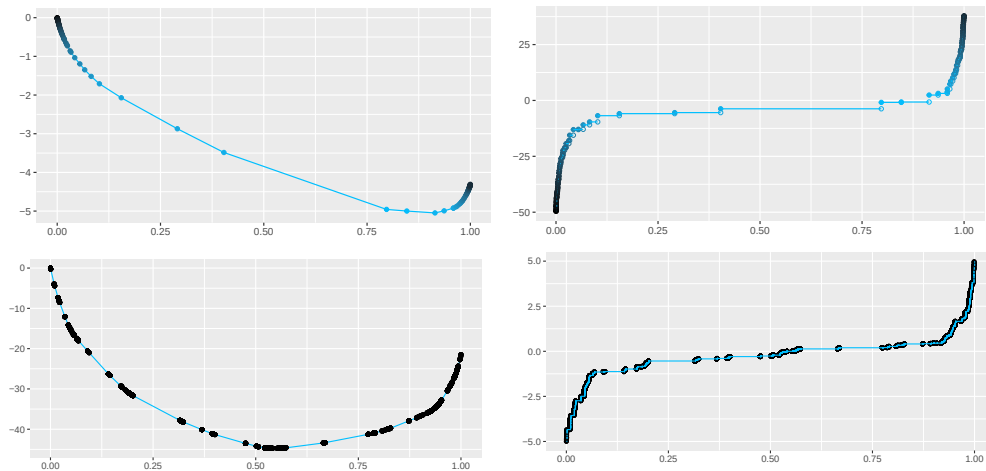


Figure: Top row: *abrupt* Lévy process with C' constant after τ_s for any $s \in \mathbb{R}$. Bottom row: *strongly eroded* Lévy process with non-constant C' after τ_s for any $s \in \mathbb{R}$.

Convex minorant C , its derivative C' and vertex time τ_s (infinite variation)

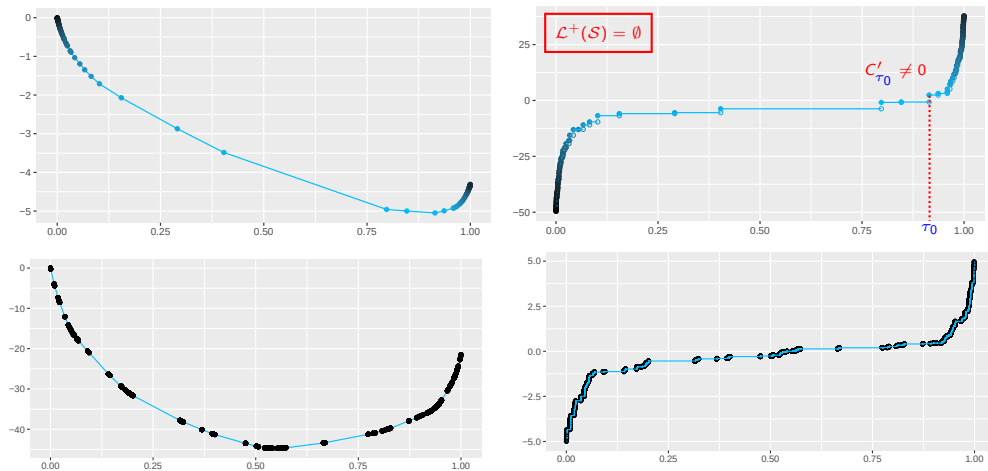


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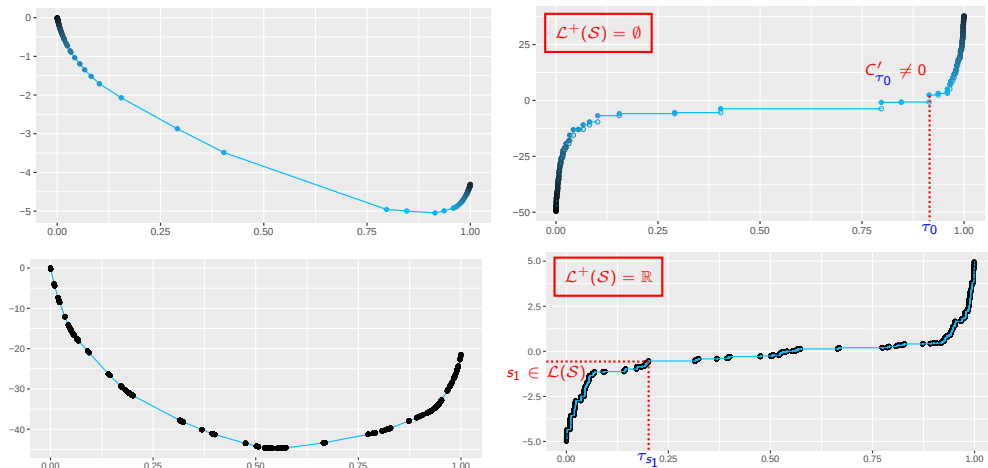


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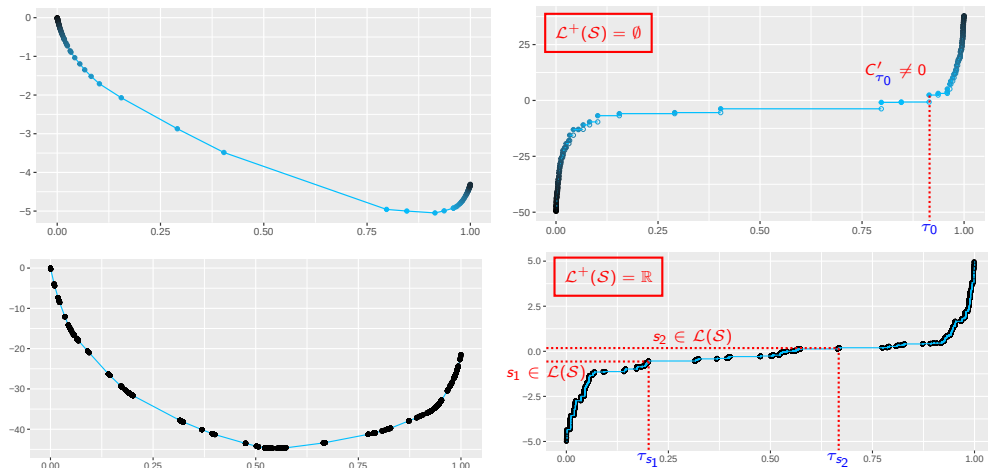


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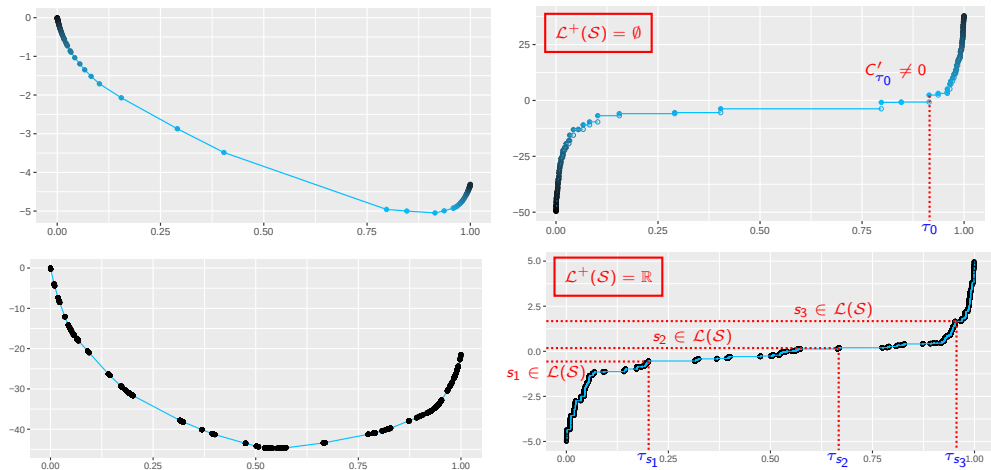


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Main questions

Regimes:	Finite slope (FS)	
Setting:	$s \in \mathcal{L}^+(\mathcal{S})$, i.e. C' non-constant at vertex time τ_s	
Upper functions:	$\limsup_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$	
Lower functions:	$\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$	

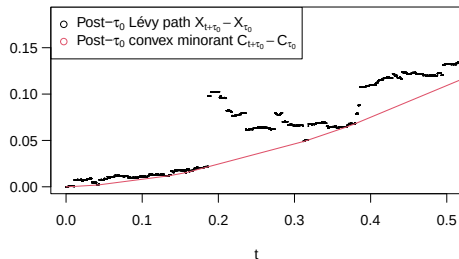


Figure: (FS) regime.

Can the results from (FS) can be applied to find lower functions of the *Lévy meander* at zero?

Main questions

Regimes:	Finite slope (FS)	Infinite slope (IS)
Setting:	$s \in \mathcal{L}^+(\mathcal{S})$, i.e. C' non-constant at vertex time τ_s	IV X , i.e. $\lim_{t \downarrow 0} C'_t = -\infty$ and non-constant C' at time 0
Upper functions:	$\limsup_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$	$\limsup_{t \downarrow 0} C'_t f(t)$
Lower functions:	$\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$	$\liminf_{t \downarrow 0} C'_t f(t)$

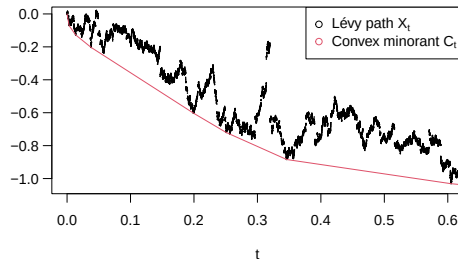
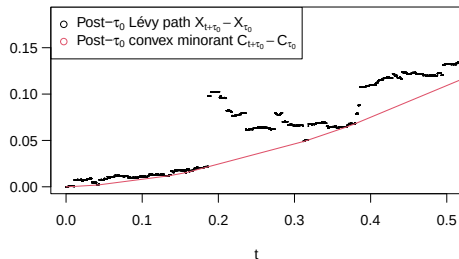


Figure: Left: (FS) regime. Right: (IS) regime.

Can the results from (FS) can be applied to find lower functions of the *Lévy meander* at zero?

Regime **(FS)**: **lower** functions at vertex time τ_s

Let $c > 0$ and consider the following conditions:

$$\int_0^1 \mathbb{P}(0 < (X_t - st)/t \leq f(t/c)) \frac{dt}{t} < \infty, \quad (1)$$

$$\int_0^1 \mathbb{E} \left[\frac{t}{f^{-1}((X_t - st)/t)^2} \mathbb{1}_{\{f(t/2) < (X_t - st)/t \leq 1\}} \right] dt < \infty, \quad (2)$$

$$2^n \int_0^{2^{-n}} \mathbb{P}(f(t/2) < (X_t - st)/t \leq f(2^{-n})) dt \xrightarrow{n \rightarrow \infty} 0. \quad (3)$$

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Theorem 2.1

Let $s \in \mathcal{L}^+(\mathcal{S})$ and f be continuous and increasing with $f(0) = 0 = \lim_{c \downarrow 0} \limsup_{t \downarrow 0} f(ct)/f(t)$ and $f(t) \leq 1$ for $t \in (0, 1]$. Then:

- (i) If (1)–(3) hold for $c = 1$, then $\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) = \infty$ a.s.
- (ii) If (1) fails for every $c > 0$, then $\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) = 0$ a.s.
- (iii) If $\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) > 1$ a.s., then (1) holds for all $c > 1$.

Regime **(FS)**: **lower** functions at vertex time τ_s

A simple characterisation of lower functions for the stable process:

Corollary 2.2

Let X be an α -stable Lévy process with $\alpha \in (0, 1]$, and let $s \in \mathcal{L}^+(\mathcal{S})$ and f have a regularly varying derivative $f' > 0$ with positive index. If $\int_0^1 t^{-1/\alpha} f(t) dt$ is finite (resp. infinite), then $\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$ equals ∞ (resp. 0) a.s.

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Remark.

Since $\int_0^1 t^{-1/\alpha} f(t) dt$ is either finite or infinite, $\liminf_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t)$ cannot be in $(0, \infty)$.

Regime (FS): upper functions at vertex time τ_s

If $\alpha < 1$, then $X \in \mathcal{Z}_{\alpha,\rho}$ if $((X_{ut} - \gamma_0 ut)/g(t))_{u \in [0,T]} \xrightarrow{d} (Z_u)_{u \in [0,T]}$ as $t \downarrow 0$ for regular varying g at 0, $\rho = \mathbb{P}(Z_1 > 0) \in [0, 1]$ and $\gamma_0 = \lim_{t \downarrow 0} X_t/t$. Let $G(t) = t/g(t)$ for $t > 0$.

Theorem 2.3

Let $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (0, 1)$ and $\rho \in (0, 1]$. Let $f : (0, 1) \rightarrow \mathbb{R}_+$ be given by $f(t) := 1/G(t(\log t^{-1})^p)$, for some $p > 0$ and all $t \in (0, 1)$. Then the following statements hold for $s = \gamma_0$:

- (i) if $p > 1/\rho$, then $\limsup_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) = 0$ a.s.,
- (ii) if $p < 1/\rho$, then $\limsup_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) = \infty$ a.s.

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- (ii) if $p < 1/\rho$, then $\limsup_{t \downarrow 0} (C'_{t+\tau_s} - s)/f(t) = \infty$ a.s.

The function f is regularly varying at 0 with index $1/\alpha - 1$. If $g(t) = t^{1/\alpha}$, then $f(t) = t^{1/\alpha-1}(\log t^{-1})^{p(1/\alpha-1)}$.

Regime (IS): upper functions at time 0

Let $c > 0$, denote $F(t) := t/f(t)$ for $t > 0$ and consider the following conditions:

$$\int_0^1 \mathbb{P}(X_t \leq -cF(t)) \frac{dt}{t} < \infty, \quad (4)$$

$$\int_0^1 \mathbb{E}[(X_t/t)^2 \mathbb{1}_{\{-2F(t) < X_t \leq -t\}}] f(t)^2 \frac{dt}{t} < \infty, \quad (5)$$

$$2^n \int_0^{2^{-n}} \mathbb{P}(-t/f(2^{-n}) \geq X_t > -2F(t/2)) dt \xrightarrow{n \rightarrow \infty} 0. \quad (6)$$

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Theorem 2.4

Let f be continuous and increasing with $f(0) = 0 = \lim_{c \downarrow 0} \limsup_{t \downarrow 0} f(ct)/f(t)$ and $f(t) \leq 1$ for $t \in (0, 1]$. Then:

- (i) If (4)–(6) hold for $c = 1$ and f is concave, then $\limsup_{t \downarrow 0} |C'_t|f(t) = 0$ a.s.
- (ii) If (4) fails for all $c > 0$, then $\limsup_{t \downarrow 0} |C'_t|f(t) = \infty$ a.s.
- (iii) If $\limsup_{t \downarrow 0} |C'_t|f(t) < 1$ a.s., then (4) holds for all $c > 1$.

Regime (IS): upper functions at time 0

Simple and sufficient conditions for α -stable law.

Corollary 2.5

Let X be an α -stable process with $\alpha \geq 1$, then:

- (a) If $t \mapsto t^{-1/\alpha} F(t)$ is bounded as $t \downarrow 0$, then $\limsup_{t \downarrow 0} |C'_t| f(t) = \infty$ a.s.
- (b) If $t^{-1/\alpha} F(t) \rightarrow \infty$ as $t \downarrow 0$ and X is not spectrally positive, then the limit $\limsup_{t \downarrow 0} |C'_t| f(t)$ is equal to ∞ (resp. 0) a.s. if the integral $\int_0^1 F(t)^{-\alpha} dt$ is infinite (resp. finite).

Regime (IS): lower functions at time 0

If $\alpha > 1$, then $X \in \mathcal{Z}_{\alpha, \rho}$ if $(X_{ut}/g(t))_{u \in [0, T]} \xrightarrow{d} (Z_u)_{u \in [0, T]}$ as $t \downarrow 0$ for regular varying g at 0 and $\rho = \mathbb{P}(Z_1 > 0) \in [0, 1]$.

Theorem 2.6

Let $X \in \mathcal{Z}_{\alpha, \rho}$ for some $\alpha \in (1, 2]$ and $\rho \in (0, 1)$. Let $f : (0, 1) \rightarrow \mathbb{R}_+$ be given by $f(t) := G(t(\log t^{-1})^p)$, for some $p > 0$ and all $t \in (0, 1)$. Then:

- (i) if $p > 1/(1 - \rho)$, then $\liminf_{t \downarrow 0} |C'_t| f(t) = \infty$ a.s.,
- (ii) if $p < 1/(1 - \rho)$, then $\liminf_{t \downarrow 0} |C'_t| f(t) = 0$ a.s.

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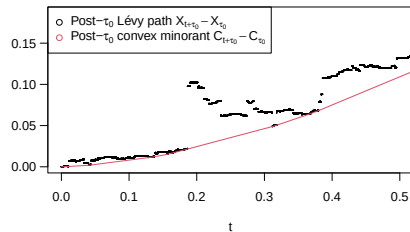
The function f is regularly varying at 0 with index $1 - 1/\alpha$. In particular, if $g(t) = t^{1/\alpha}$, we obtain $f(t) = t^{1-1/\alpha}(\log t^{-1})^{p(1-1/\alpha)}$.

Regime (FS): Lower functions of the Lévy path at vertex times

Let $0 \in \mathcal{L}^+(\mathcal{S})$, $m_0 := \inf_{t \in [0, T]} X_t$, $f : [0, \infty) \rightarrow [0, \infty)$ non-decreasing and $\phi(t) := \int_0^t f(u) du$, $t \geq 0$. Then:

$$* \liminf_{t \downarrow 0} \frac{C'_{t+\tau_0}}{f(t)} = \infty \implies \lim_{t \downarrow 0} \frac{X_{t+\tau_0} - m_0}{\phi(t)} = \infty,$$

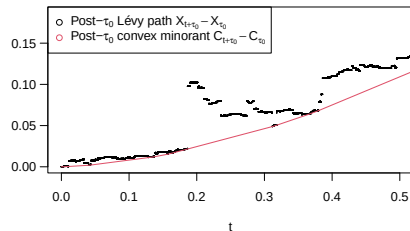
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- * $\liminf_{t \downarrow 0} \frac{C'_{t+\tau_0}}{f(t)} = \infty \implies \lim_{t \downarrow 0} \frac{X_{t+\tau_0} - m_0}{\phi(t)} = \infty$,
- * $\limsup_{t \downarrow 0} \frac{C'_{t+\tau_0}}{f(t)} = 0 \implies \liminf_{t \downarrow 0} \frac{X_{t+\tau_0} - m_0}{\phi(t)} = 0$.



Corollary 2.7

Let X be an α -stable process for $\alpha \in (0, 1)$ and $\rho \in (0, 1]$. Given $p \in \mathbb{R}$, then $\phi(t) \sim \alpha \log^p(1/t) t^{1/\alpha}$ as $t \downarrow 0$ and:

- (i) If $p > (1/\alpha - 1)/\rho$, then $\liminf_{t \downarrow 0} (X_{t+\tau_0} - m_0)/\phi(t) = 0$ a.s.
- (ii) If $\alpha \in (1/2, 1)$ and $p < -1$ then $\lim_{t \downarrow 0} (X_{t+\tau_0} - m_0)/\phi(t) = \infty$ a.s.
- (iii) If $\alpha \in (0, 1/2]$, then $\lim_{t \downarrow 0} (X_{t+\tau_0} - m_0)/t^q = \infty$ a.s. for any $q > 1/\alpha \geq 2$.

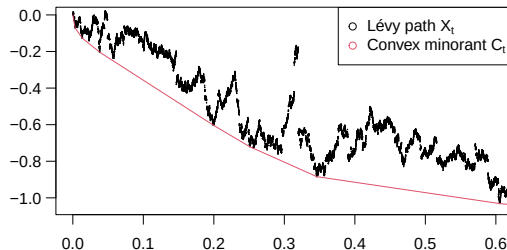
Regime (IS): Upper and lower function of the Lévy path at vertex times

Recall that in regime (IS) we assume that X is of infinite variation.

Lemma 2.8

Let the function $f : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing with $f(0) = 0$ and finite $\tilde{f}(t) := \int_0^t f(u)^{-1} du$, $t \geq 0$. Then:

- (i) Assume that $\limsup_{t \downarrow 0} |C'_t| f(t) = 0$ a.s., then we have $\limsup_{t \downarrow 0} (-X_t)/\tilde{f}(t) = 0$ a.s.
- (ii) Assume that $\liminf_{t \downarrow 0} |C'_t| f(t) = \infty$ a.s. and $\tilde{f}$ is concave on a neighborhood of 0, then we have $\limsup_{t \downarrow 0} (-X_t)/\tilde{f}(t) = \infty$ a.s.



Proofs: additive process with increasing paths

$Y = (Y_t)_{t \geq 0}$, with $Y_0 = 0$, is pure-jump right-continuous non-decreasing additive process (i.e. with independent non-stationary increments), mean jump measure $\Pi(dt, dx)$ for $(t, x) \in [0, \infty) \times (0, \infty)$ and its Laplace exponent

$$\mathbb{E}[e^{-uY_t}] = e^{-\Psi_t(u)}, \quad \text{where} \quad \Psi_t(u) := \int_{\mathbb{R}_+} (1 - e^{-ux}) \Pi((0, t], dx), \quad \text{for any } u \geq 0.$$

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Let $L_t := \inf\{u > 0 : Y_u > t\}$, $t \geq 0$, denote the right-continuous inverse of Y .

Upper/lower functions of L_t require control on $\Psi_t(u)$ as $u \rightarrow \infty$ and $t \rightarrow 0$.

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* Regime (IS): $Y_v := \tau_{-1/v}$, $v > 0$, yields $C'_t = -1/L_t$, $t > 0$.

(See details in [1].)

- [1] David Bang, Jorge González Cázares, and Aleksandar Mijatović. “How smooth can the convex hull of a Lévy process be?” (2022).
- [2] David Bang, Jorge González Cázares, and Aleksandar Mijatović. “When is the convex minorant of a Lévy process smooth?” (2022). [arXiv: 2205.14416v2 \[math.PR\]](#).

Thank you for watching!