

When is the convex hull of a Lévy path smooth?

David Bang



Jorge González Cázares



Aleksandar Mijatović



University of Warwick & The Alan Turing Institute

May 25, 2022

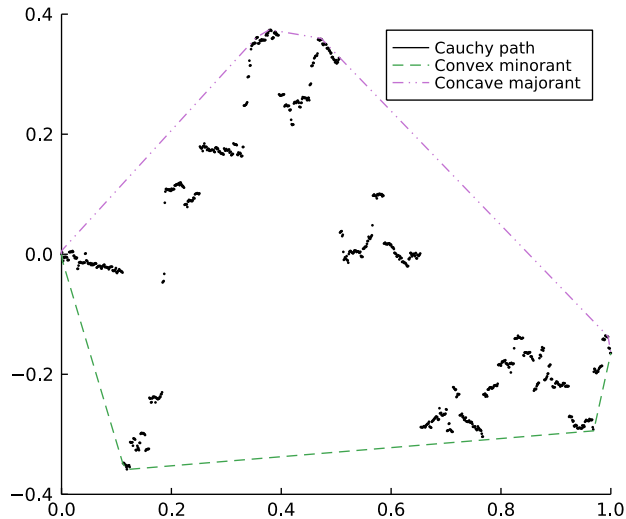


Figure: The path of a Cauchy process on $[0, 1]$ and its convex hull [2].

Piecewise linear convex function and associated set of slopes

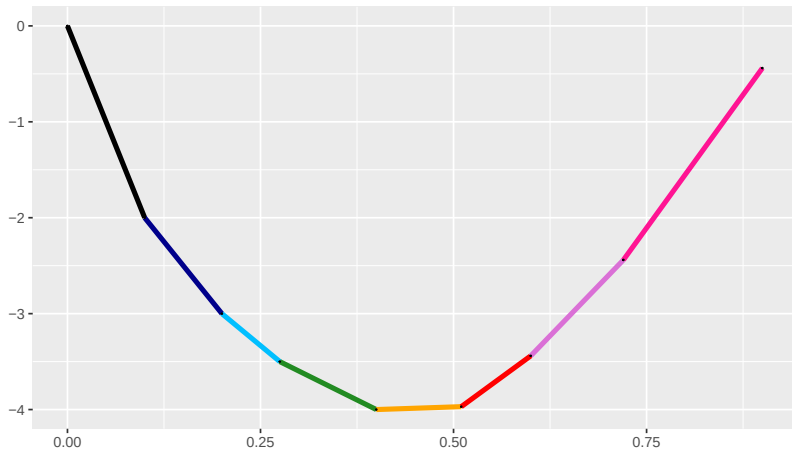


Figure: Piecewise linear convex function C (8 faces)

Piecewise linear convex function and associated set of slopes

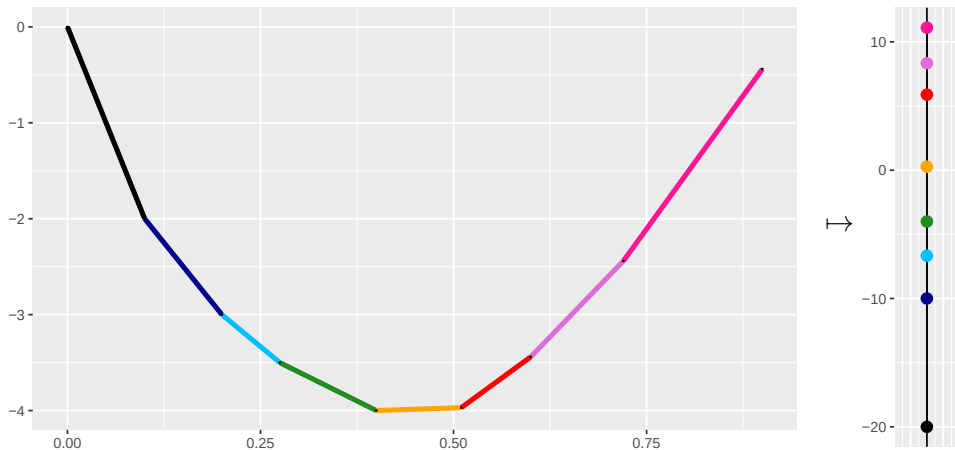


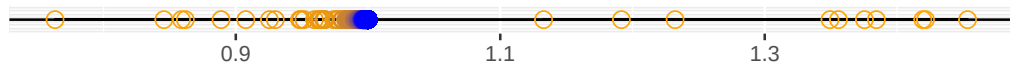
Figure: Piecewise linear convex function C (8 faces)
and corresponding set of slopes S

Definitions

- * Convex minorant C of a path of X is piecewise linear [3] with countably many faces over maximal subintervals (a_n, b_n) of $[0, T]$ with $\sum_{n \in \mathbb{N}} (b_n - a_n) = T$.
- * Let C' denote the right-derivative of C . Note C' is constant on each (a_n, b_n) .
- * Let $\mathcal{S}$ be the set of slopes of the faces of C (i.e. $\mathcal{S} = \{C'((a_n, b_n)) : n \in \mathbb{N}\}$).

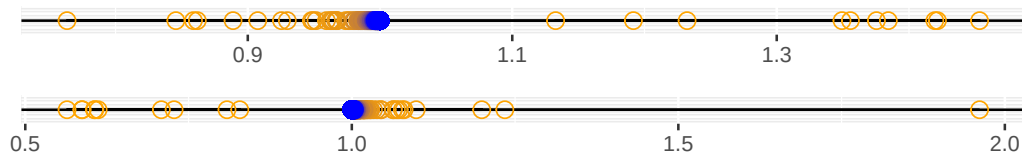
Left, right and two-sided accumulation points of $\mathcal{S}$

Denote by $\mathcal{L}^-(\mathcal{S})$ (resp. $\mathcal{L}^+(\mathcal{S})$) the set of all left (resp. right) limit points of $\mathcal{S} \subset \mathbb{R}$.



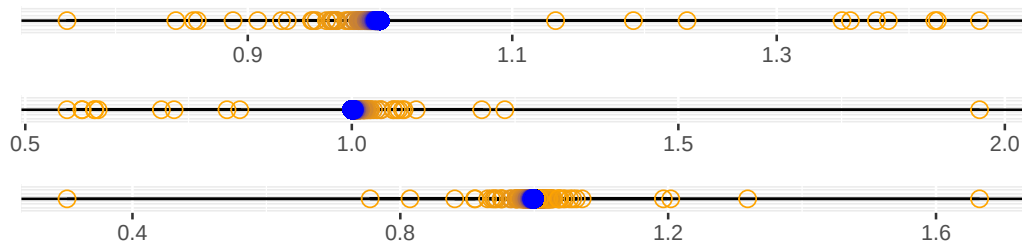
Left, right and two-sided accumulation points of $\mathcal{S}$

Denote by $\mathcal{L}^-(\mathcal{S})$ (resp. $\mathcal{L}^+(\mathcal{S})$) the set of all left (resp. right) limit points of $\mathcal{S} \subset \mathbb{R}$.



Left, right and two-sided accumulation points of $\mathcal{S}$

Denote by $\mathcal{L}^-(\mathcal{S})$ (resp. $\mathcal{L}^+(\mathcal{S})$) the set of all left (resp. right) limit points of $\mathcal{S} \subset \mathbb{R}$.



Let $\mathcal{L}(\mathcal{S}) := \mathcal{L}^-(\mathcal{S}) \cup \mathcal{L}^+(\mathcal{S})$ be the (closed) set of all limit points of $\mathcal{S}$.

Theorem 1

For any measurable set $I \subseteq \mathbb{R}$, the set $\mathcal{S} \cap I$ is either a.s. finite or a.s. infinite. Moreover, the cardinality $|\mathcal{S} \cap I|$ of the intersection $\mathcal{S} \cap I$ is infinite a.s. if and only if

$$\int_0^1 \mathbb{P}(X_t/t \in I) \frac{dt}{t} = \infty. \quad (1)$$

Theorem 1

For any measurable set $I \subseteq \mathbb{R}$, the set $\mathcal{S} \cap I$ is either a.s. finite or a.s. infinite. Moreover, the cardinality $|\mathcal{S} \cap I|$ of the intersection $\mathcal{S} \cap I$ is infinite a.s. if and only if

$$\int_0^1 \mathbb{P}(X_t/t \in I) \frac{dt}{t} = \infty. \quad (1)$$

Theorem 2

The boundary of the convex hull of the graph $t \mapsto (t, X_t)$, $t \in [0, T]$, of a path of any Lévy process X is continuously differentiable (as a closed curve in $\mathbb{R}^2$) a.s. if and only if (1) holds for all intervals I in $\mathbb{R}$. Moreover, this is equivalent to the set $\mathcal{S}$ being dense in $\mathbb{R}$ a.s.

Theorem 1

For any measurable set $I \subseteq \mathbb{R}$, the set $\mathcal{S} \cap I$ is either a.s. finite or a.s. infinite. Moreover, the cardinality $|\mathcal{S} \cap I|$ of the intersection $\mathcal{S} \cap I$ is infinite a.s. if and only if

$$\int_0^1 \mathbb{P}(X_t/t \in I) \frac{dt}{t} = \infty. \quad (1)$$

Theorem 2

The boundary of the convex hull of the graph $t \mapsto (t, X_t)$, $t \in [0, T]$, of a path of any Lévy process X is continuously differentiable (as a closed curve in $\mathbb{R}^2$) a.s. if and only if (1) holds for all intervals I in $\mathbb{R}$. Moreover, this is equivalent to the set $\mathcal{S}$ being dense in $\mathbb{R}$ a.s.

Corollary 1

The limit sets $\mathcal{L}(\mathcal{S})$ and $\mathcal{L}^\pm(\mathcal{S})$ of $\mathcal{S}$ are deterministic a.s.

Finite variation X – results

Let $\psi(u) = \log \mathbb{E} e^{iuX_1} = iu\gamma_0 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)$ be the Lévy-Khintchine exponent of X .

Finite variation X – results

Let $\psi(u) = \log \mathbb{E} e^{iuX_1} = iu\gamma_0 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)$ be the Lévy-Khintchine exponent of X .

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
Finite variation (FV)	C' bounded below <i>and</i> above; C' discontinuous on boundary ∂I_r , $\forall r \in \mathcal{S}; \mathcal{L}(\mathcal{S}) = \{\gamma_0\}$, where $\gamma_0 = \lim_{t \downarrow 0} X_t/t$ a.s., and $\gamma_0 \notin \mathcal{S}$	atomic; atoms accumulate from left/right or both sides at a unique (random) limit point in $[0, T]$

Finite variation X – results

Let $\psi(u) = \log \mathbb{E} e^{iuX_1} = iu\gamma_0 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)$ be the Lévy-Khintchine exponent of X .

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
Finite variation (FV)	C' bounded below <i>and</i> above; C' discontinuous on boundary ∂I_r , $\forall r \in \mathcal{S}; \mathcal{L}(\mathcal{S}) = \{\gamma_0\}$, where $\gamma_0 = \lim_{t \downarrow 0} X_t/t$ a.s., and $\gamma_0 \notin \mathcal{S}$	atomic; atoms accumulate from left/right or both sides at a unique (random) limit point in $[0, T]$

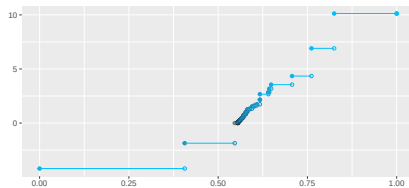
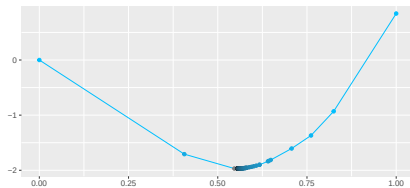
Behaviour depends on $I_- := \int_0^1 \mathbb{P}(X_t/t < \gamma_0) \frac{dt}{t}$, $I_+ := \int_0^1 \mathbb{P}(X_t/t > \gamma_0) \frac{dt}{t}$ via

$$\mathcal{L}^\pm(\mathcal{S}) = \{\gamma_0\} \xLeftrightarrow{\text{Thm 1}} I_\pm = \infty \xLeftrightarrow{[1]} \int_{(-1,1)} \frac{\max\{\pm x, 0\}}{\int_0^{\max\{\pm x, 0\}} \bar{\nu}_\mp(y) dy} \nu(dx) = \infty,$$

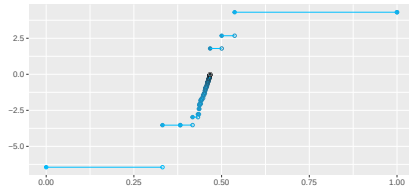
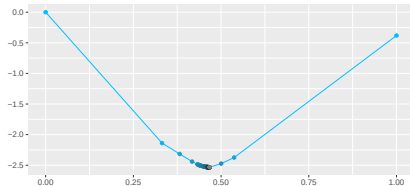
where $\bar{\nu}_+(x) := \nu((x, \infty))$ & $\bar{\nu}_-(x) := \nu((-\infty, -x))$, $x > 0$, and $\mp := -(\pm)$.

Convex minorant C (of an FV Lévy process) and its derivative C'

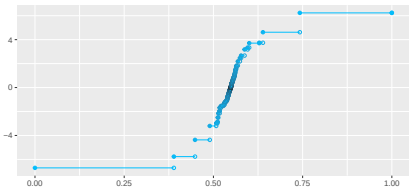
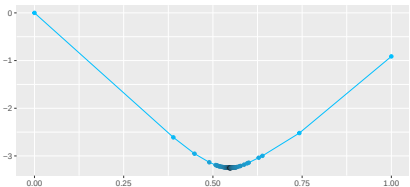
$$I_- < \infty, I_+ = \infty$$



$$I_+ < \infty, I_- = \infty$$



$$I_+ = I_- = \infty$$



Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
$\mathfrak{s}_1 \in L^1_{\text{loc}}(r),$ $\forall r \in \mathbb{R}$	C' discontinuous on boundary $\partial I_r,$ $\forall r \in \mathcal{S};$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \emptyset$	atomic; atoms accumulate only at 0 and T

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
$\mathfrak{s}_1 \in L^1_{\text{loc}}(r),$ $\forall r \in \mathbb{R}$	C' discontinuous on boundary $\partial I_r,$ $\forall r \in \mathcal{S};$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \emptyset$	atomic; atoms accumulate only at 0 and T
$\mathfrak{s}_1(r) = \infty,$ $\forall r \in \mathbb{R}$	C' is continuous on $(0, T);$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \mathbb{R}$	singular continuous

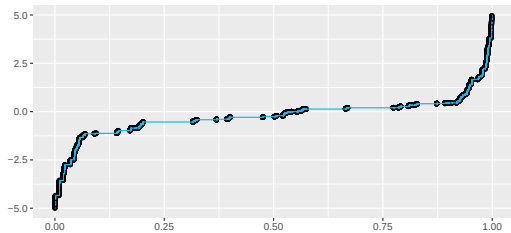
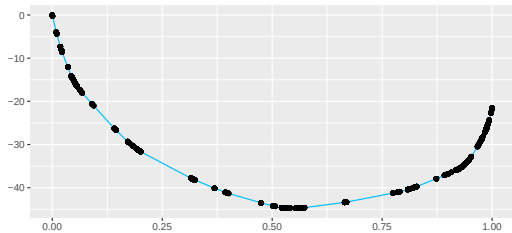
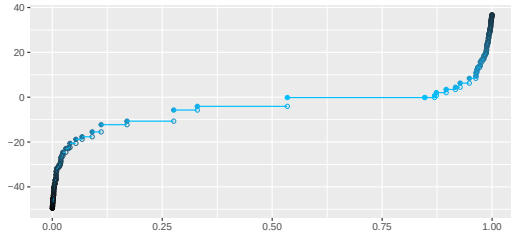
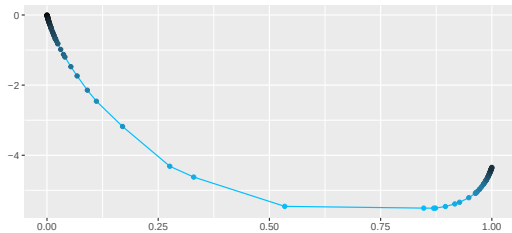


Figure: X of infinite variation (IV). Left: C ; right: C'

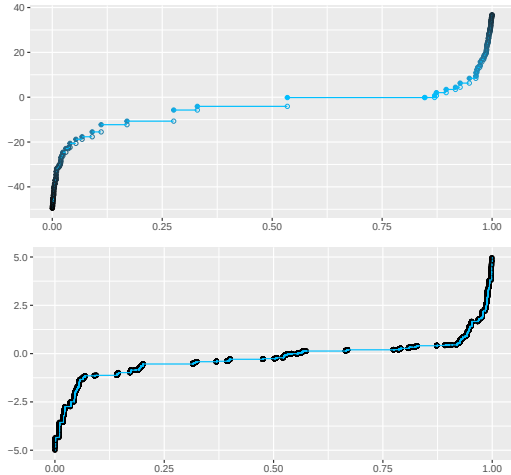
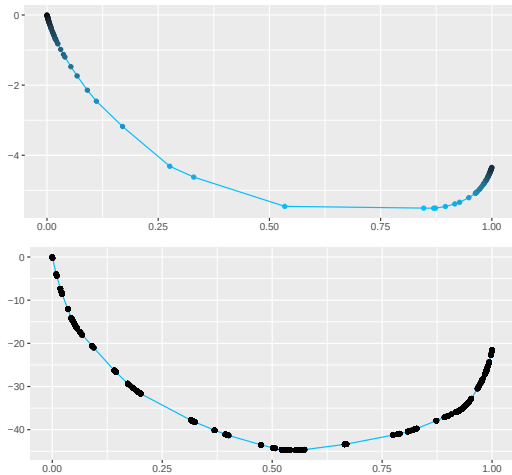


Figure: X of infinite variation (IV). Left: C ; right: C'

How can we tell which case it is in terms of the local behaviour of the paths of X ?

Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:

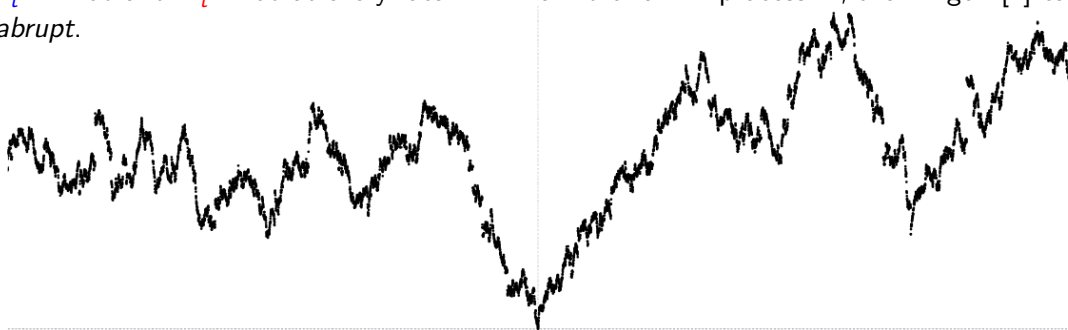
$$D_t^\uparrow := \limsup_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_{t-})/\varepsilon \text{ and } D_t^\downarrow := \liminf_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_t)/\varepsilon.$$

Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:

$$D_t^\uparrow := \limsup_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_{t-})/\varepsilon \text{ and } D_t^\downarrow := \liminf_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_t)/\varepsilon.$$

If $D_t^\uparrow = -\infty$ and $D_t^\downarrow = \infty$ at every local minimum t of an IV process X , then Vigon [4] calls X *abrupt*.



Proposition 1

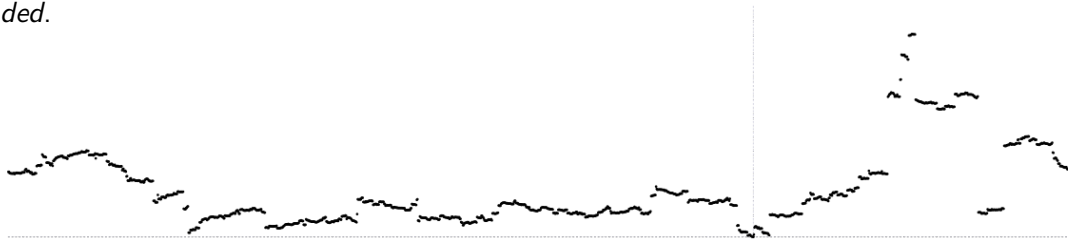
A Lévy process X is abrupt if and only if $\mathcal{L}(S) = \emptyset$ a.s.

Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:

$$D_t^\uparrow := \limsup_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_{t-})/\varepsilon \text{ and } D_t^\downarrow := \liminf_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_t)/\varepsilon.$$

If $D_t^\uparrow = 0$ and $D_t^\downarrow = 0$ at every local minimum t of an IV process X , then Vigon [4] calls X *eroded*.

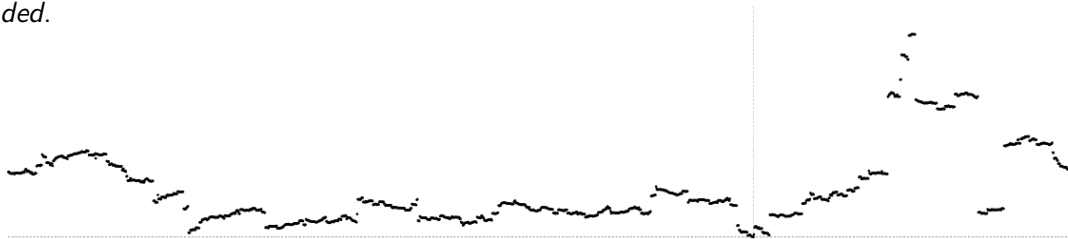


Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:

$$D_t^{\uparrow} := \limsup_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_{t-})/\varepsilon \text{ and } D_t^{\downarrow} := \liminf_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_t)/\varepsilon.$$

If $D_t^{\uparrow} = 0$ and $D_t^{\downarrow} = 0$ at every local minimum t of an IV process X , then Vigon [4] calls X *eroded*.



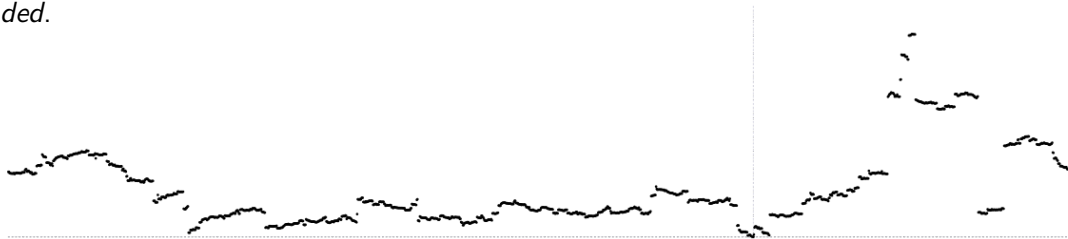
X is said to be *strongly eroded* if: $(X_t - rt)_{t \geq 0}$ is eroded $\forall r \in \mathbb{R}$.

Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:

$$D_t^\uparrow := \limsup_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_{t-})/\varepsilon \text{ and } D_t^\downarrow := \liminf_{\varepsilon \downarrow 0} (X_{t+\varepsilon} - X_t)/\varepsilon.$$

If $D_t^\uparrow = 0$ and $D_t^\downarrow = 0$ at every local minimum t of an IV process X , then Vigon [4] calls X *eroded*.



X is said to be *strongly eroded* if: $(X_t - rt)_{t \geq 0}$ is eroded $\forall r \in \mathbb{R}$.

Proposition 2

A Lévy process X is strongly eroded if and only if $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s.

A conjectural dichotomy for IV Lévy processes

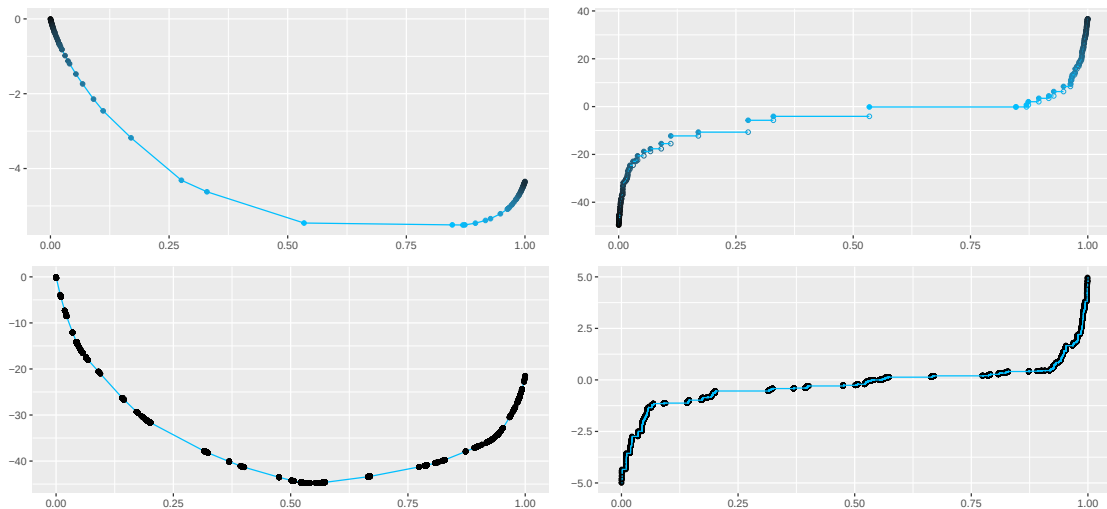


Figure: Convex minorant and its right derivative of an IV Lévy process X

A conjectural dichotomy

Conjecture 1

Any IV Lévy process is either A or SE. Equivalently, either $\mathfrak{s}_1 \in L^1_{\text{loc}}(r), \forall r \in \mathbb{R}$, or $\mathfrak{s}_1 = \infty$ a.e.

Geometrically, either the Lévy process shoots away from the convex minorant as soon as it touches it, or it stays close to the convex minorant when it touches it. This is implied by Vigon's point-hitting conjecture:

A conjectural dichotomy

Conjecture 1

Any IV Lévy process is either A or SE. Equivalently, either $\mathfrak{s}_1 \in L^1_{\text{loc}}(r), \forall r \in \mathbb{R}$, or $\mathfrak{s}_1 = \infty$ a.e.

Geometrically, either the Lévy process shoots away from the convex minorant as soon as it touches it, or it stays close to the convex minorant when it touches it. This is implied by Vigon's point-hitting conjecture:

Conjecture 2 ([5, Conject. 1.6])

Let X be an infinite variation process and for any $r \in \mathbb{R}$ define the Lévy process $X^{(r)} = (X_t - rt)_{t \geq 0}$. Then the following statements are equivalent.

- (i) There exists some $r \in \mathbb{R}$ such that the process $X^{(r)}$ hits points.*
- (ii) For all $r \in \mathbb{R}$ the process $X^{(r)}$ hits points.*
- (iii) The process X is abrupt.*

A conjectural dichotomy

Conjecture 1

Any IV Lévy process is either A or SE. Equivalently, either $\mathfrak{s}_1 \in L^1_{\text{loc}}(r), \forall r \in \mathbb{R}$, or $\mathfrak{s}_1 = \infty$ a.e.

Geometrically, either the Lévy process shoots away from the convex minorant as soon as it touches it, or it stays close to the convex minorant when it touches it. This is implied by Vigon's point-hitting conjecture:

Conjecture 2 ([5, Conject. 1.6])

Let X be an infinite variation process and for any $r \in \mathbb{R}$ define the Lévy process $X^{(r)} = (X_t - rt)_{t \geq 0}$. Then the following statements are equivalent.

- (i) There exists some $r \in \mathbb{R}$ such that the process $X^{(r)}$ hits points. ($\iff \mathfrak{s}_1(r) < \infty$.)*
- (ii) For all $r \in \mathbb{R}$ the process $X^{(r)}$ hits points. ($\iff \mathfrak{s}_1(r) < \infty, \forall r \in \mathbb{R}$.)*
- (iii) The process X is abrupt. ($\iff \mathfrak{s}_1 \in L^1_{\text{loc}}(r), \forall r \in \mathbb{R}$.)*

- [1] Jean Bertoin. “Regularity of the half-line for Lévy processes”. *Bull. Sci. Math.* 121.5 (1997), pp. 345–354. ISSN: 0007-4497.
- [2] Jean Bertoin. “The convex minorant of the Cauchy process”. *Electron. Comm. Probab.* 5 (2000), pp. 51–55. ISSN: 1083-589X. DOI: 10.1214/ECP.v5-1017. URL: <https://doi.org/10.1214/ECP.v5-1017>.
- [3] Jorge Ignacio González Cázares and Aleksandar Mijatović. “Convex minorants and the fluctuation theory of Lévy processes”. (2021). arXiv: 2105.15060.
- [4] Vincent Vigon. “Abrupt Lévy processes”. *Stochastic Process. Appl.* 103.1 (2003), pp. 155–168. ISSN: 0304-4149. DOI: 10.1016/S0304-4149(02)00186-2. URL: [https://doi.org/10.1016/S0304-4149\(02\)00186-2](https://doi.org/10.1016/S0304-4149(02)00186-2).
- [5] Vincent Vigon. *Dérivées de Dini de processus de Lévy*. Retrieved 1st September, 2021, from <https://drive.google.com/file/d/0B5ZYbxojmGIadE9U0V9KR0QyT1E/view?usp=sharing&resourcekey=0-ZIypnqJtKmdvZoBpR0riTw>. 2016.

The arxiv link to the article is in the description of the video.

Thank you for watching.

When is the convex hull of a Lévy path smooth?

David Bang



Jorge González Cázares



Aleksandar Mijatović



University of Warwick & The Alan Turing Institute

May 25, 2022

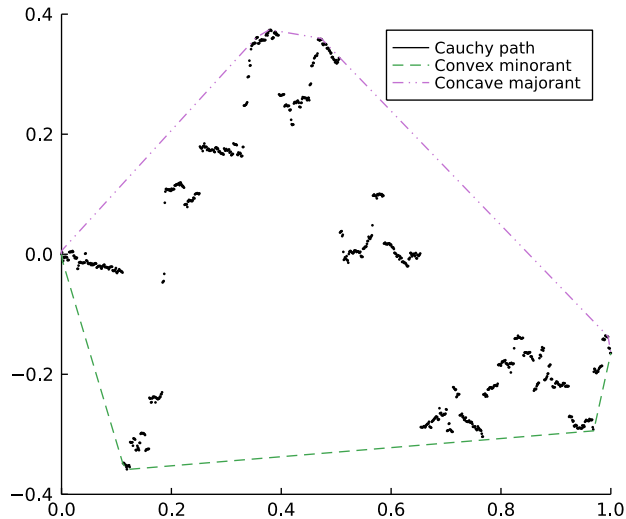


Figure: The path of a Cauchy process on $[0, 1]$ and its convex hull [2].

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
Abrupt $\mathfrak{s}_1 \in L^1_{\text{loc}}(r),$ $\forall r \in \mathbb{R}$	C' discontinuous on boundary $\partial I_r,$ $\forall r \in \mathcal{S};$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \emptyset$	atomic; atoms accumulate only at 0 and T

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$ is the Lévy-Khintchine exponent of X and define

$$\mathfrak{s}_1(r) := \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \in (0, \infty], \quad r \in \mathbb{R}.$$

Lévy process X	Derivative C' and the limit set $\mathcal{L}(\mathcal{S})$	Measure dC'
Abrupt $\mathfrak{s}_1 \in L^1_{\text{loc}}(r),$ $\forall r \in \mathbb{R}$	C' discontinuous on boundary $\partial I_r,$ $\forall r \in \mathcal{S};$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \emptyset$	atomic; atoms accumulate only at 0 and T
Strongly eroded $\mathfrak{s}_1(r) = \infty,$ $\forall r \in \mathbb{R}$	C' is continuous on $(0, T);$ $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty;$ $\mathcal{L}(\mathcal{S}) = \mathbb{R}$	singular continuous

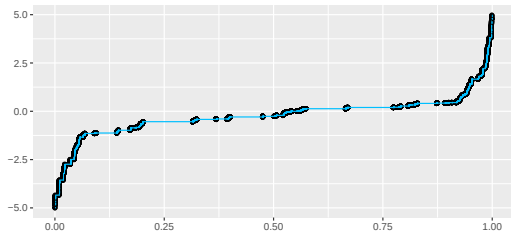
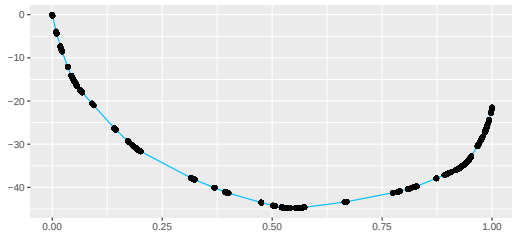
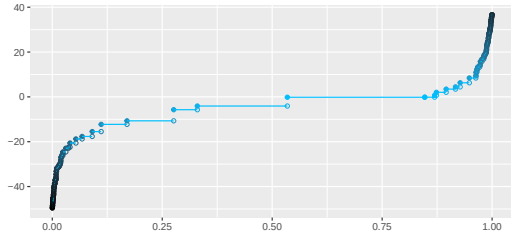
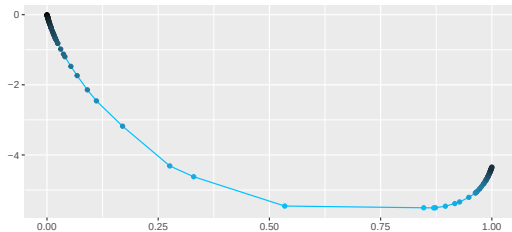


Figure: X of infinite variation (IV). Left: C ; right: C'

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$. Note $\Re \psi(u) \leq 0$, implying $\Re(1/(1 + iur - \psi(u))) > 0 \ \forall r, u \in \mathbb{R}$ and

$$0 < \mathfrak{s}_1(r) = \frac{1}{2\pi} \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \leq \infty, \quad r \in \mathbb{R}.$$

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$. Note $\Re \psi(u) \leq 0$, implying $\Re(1/(1 + iur - \psi(u))) > 0 \ \forall r, u \in \mathbb{R}$ and

$$0 < s_1(r) = \frac{1}{2\pi} \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \leq \infty, \quad r \in \mathbb{R}.$$

Connection to hitting points and local time!

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$. Note $\Re \psi(u) \leq 0$, implying $\Re(1/(1 + iur - \psi(u))) > 0 \forall r, u \in \mathbb{R}$ and

$$0 < s_1(r) = \frac{1}{2\pi} \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \leq \infty, \quad r \in \mathbb{R}.$$

Connection to hitting points and local time!

Proposition 3

$r \in \mathcal{L}(\mathcal{S}) \iff s_1 \notin L^1_{\text{loc}}(r)$ (i.e. s_1 not integrable on any neighbourhood of $r \in \mathbb{R}$).

Infinite variation X – results

Recall $\psi(u) = \log \mathbb{E} e^{iuX_1}$. Note $\Re \psi(u) \leq 0$, implying $\Re(1/(1 + iur - \psi(u))) > 0 \ \forall r, u \in \mathbb{R}$ and

$$0 < s_1(r) = \frac{1}{2\pi} \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \leq \infty, \quad r \in \mathbb{R}.$$

Connection to hitting points and local time!

Proposition 3

$r \in \mathcal{L}(\mathcal{S}) \iff s_1 \notin L^1_{\text{loc}}(r)$ (i.e. s_1 not integrable on any neighbourhood of $r \in \mathbb{R}$).

Proof.

Theorem 1 and the following equivalence (proved by Vigon via WH factorisation; our simpler argument uses Gaussian perturbation): for any real $a < b$ it holds that

$$\int_a^b s_1(r) dr < \infty \iff \int_0^1 \mathbb{P}(X_t/t \in (a, b)) \frac{dt}{t} < \infty.$$



Infinite variation X – results

The set of slopes $\mathcal{S}$ is unbounded on both sides for any X of IV, i.e. $\sup \mathcal{S} = -\inf \mathcal{S} = \infty$, and hence $-\lim_{t \downarrow 0} C'(t) = \lim_{t \uparrow T} C'(t) = \infty$ a.s.

Any SE (resp. A) process, when perturbed by a finite variation process, is still SE (resp. A).

Proposition 4

Suppose $X = Y + Z$ for (possibly dependent) Lévy processes Y and Z . Let $\mathcal{S}_X$ and $\mathcal{S}_Z$ be the sets of slopes of the faces of the convex minorants of X and Z , respectively. If Y is of FV (possibly finite activity) with natural drift b , then $\mathcal{L}(\mathcal{S}_X) = \mathcal{L}(\mathcal{S}_Z) + b$.

Recipe to construct many SE and A processes!

Too much jump activity breaks smoothness

Let $\sigma \geq 0$ be the volatility of the Brownian component of X and define the function $\bar{\sigma}^2(u) := \int_{(-u,u)} x^2 \nu(dx)$ for $u > 0$. Define the indices:

$$\text{Lower-activity index: } \beta_- := \inf \left\{ p > 0 : \liminf_{u \downarrow 0} u^{p-2} \bar{\sigma}^2(u) = 0 \right\}$$

$$\text{Blumenthal-Gettoor index: } \beta_+ := \inf \left\{ p > 0 : \int_{(-1,1)} |x|^p \nu(dx) < \infty \right\}$$

Too much jump activity breaks smoothness

Let $\sigma \geq 0$ be the volatility of the Brownian component of X and define the function $\bar{\sigma}^2(u) := \int_{(-u,u)} x^2 \nu(dx)$ for $u > 0$. Define the indices:

$$\text{Lower-activity index: } \beta_- := \inf \left\{ p > 0 : \liminf_{u \downarrow 0} u^{p-2} \bar{\sigma}^2(u) = 0 \right\}$$

$$\text{Blumenthal-Gettoor index: } \beta_+ := \inf \left\{ p > 0 : \int_{(-1,1)} |x|^p \nu(dx) < \infty \right\}$$

Proposition 5

If $\int_1^\infty (1 + u^2(\sigma^2 + \bar{\sigma}^2(1/u)))^{-1} du < \infty$, then $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. and hence X is abrupt. In particular, this is the case if $\sigma^2 > 0$ or $\beta_- > 1$.

Note that $0 \leq \beta_- \leq \beta_+ \leq 2$, and, for SE process, $\beta_- \leq 1 \leq \beta_+$.

Examples: α -stable processes with $\alpha > 1$ are abrupt.

Too much asymmetry breaks smoothness

Proposition 6

For IV X with $\nu(dx) \ll \nu(d(-x))$ on an interval $x \in (0, \varepsilon)$ with Radon-Nikodym derivative $\varphi(x)$ satisfying $\liminf_{x \downarrow 0} \varphi(x) > 1$, then $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. making X abrupt.

Examples:

- * Weakly 1-stable process, i.e. $\nu((-\infty, -x)) = c_- x^{-1}$ and $\nu((x, \infty)) = c_+ x^{-1}$ for all $x > 0$ and some $c_+ \neq c_-$.
- * Asymmetric Orey's process (discussed below).
- * More generally, if $\nu((-\infty, -x)) = c\nu((x, \infty))$ for all $x > 0$ and some $c \neq 1$.

Sufficient conditions for X to be strongly eroded (or abrupt)

Corollary 2

Let X be a Lévy process of IV with $e^{\psi(u)} = \mathbb{E}e^{iuX_1}$, $u \in \mathbb{R}$.

- (i) If $\limsup_{u \rightarrow \infty} |\psi(u)/u| < \infty$, then X is SE.
- (ii) If $\lim_{u \rightarrow \infty} |\psi(u)/u| = \infty$, then X is either A or SE.

Most Lévy processes are included! Examples:

- (a) Any process attracted to an α -stable process in small-time with $\alpha \in (1, 2]$;
- (b) We can tell in (ii) whether X is SE or A, e.g. (a) is in A;
- (c) 1-semi-stable processes is SE if it is strictly 1-semi-stable and otherwise it is A.

Sufficient conditions for X to be strongly eroded (or abrupt)

Corollary 2

Let X be a Lévy process of IV with $e^{\psi(u)} = \mathbb{E}e^{iuX_1}$, $u \in \mathbb{R}$.

- (i) If $\limsup_{u \rightarrow \infty} |\psi(u)/u| < \infty$, then X is SE.
- (ii) If $\lim_{u \rightarrow \infty} |\psi(u)/u| = \infty$, then X is either A or SE.

Most Lévy processes are included! Examples:

- (a) Any process attracted to an α -stable process in small-time with $\alpha \in (1, 2]$;
- (b) We can tell in (ii) whether X is SE or A, e.g. (a) is in A;
- (c) 1-semi-stable processes is SE if it is strictly 1-semi-stable and otherwise it is A.

It excludes Lévy processes with $\liminf_{|u| \rightarrow \infty} |\psi(u)/u| < \infty = \limsup_{|u| \rightarrow \infty} |\psi(u)/u|$.

Examples: Orey's process (see two slides on) and other SE processes (see three slides on).

Domain of normal attraction to Cauchy process

It is known that the convex hull of a Cauchy process is SE. Similarly, processes in the domain of *normal* attraction are also SE:

Example 9.1

- * Suppose $X_t/t \xrightarrow{d} S$ as $t \downarrow 0$ for some Cauchy random variable S , then X is SE.
- * Indeed, $\lim_{t \downarrow 0} \mathbb{P}(X_t/t \in (a, b)) = \mathbb{P}(S \in (a, b))$ for any $a < b$ by assumption, so Theorem 1 immediately gives the claim.
- * Such an assumption is satisfied if $\nu([x, \infty))x \rightarrow c$, $\nu((-\infty, -x])x \rightarrow c$ and $\int_{(-1, -x] \cup [x, 1)} y \nu(dy) \rightarrow c'$ as $x \downarrow 0$ for some $c > 0$ and $c' \in \mathbb{R}$.

There exist abrupt processes attracted to Cauchy process.

Orey's process

The following Lévy process satisfies $\liminf_{u \rightarrow \infty} |\psi(u)/u| < \infty$ and $\limsup_{u \rightarrow \infty} |\psi(u)/u| = \infty$, making Corollary 2 inapplicable.

Orey's process

The following Lévy process satisfies $\liminf_{u \rightarrow \infty} |\psi(u)/u| < \infty$ and $\limsup_{u \rightarrow \infty} |\psi(u)/u| = \infty$, making Corollary 2 inapplicable.

Example 9.2

- * Fix any $\alpha \in (1, 2)$, $c_{\pm} \geq 0$ with $c_+ + c_- > 0$ and integer $\eta > 2/(2 - \alpha)$. Set $\sigma = 0$, $b = 0$ and $\nu = \sum_{n \in \mathbb{N}} a_n^{-\alpha} (c_+ \delta_{a_n} + c_- \delta_{-a_n})$ for $a_n = 2^{-\eta^n}$.

Orey's process

The following Lévy process satisfies $\liminf_{u \rightarrow \infty} |\psi(u)/u| < \infty$ and $\limsup_{u \rightarrow \infty} |\psi(u)/u| = \infty$, making Corollary 2 inapplicable.

Example 9.2

- * Fix any $\alpha \in (1, 2)$, $c_{\pm} \geq 0$ with $c_+ + c_- > 0$ and integer $\eta > 2/(2 - \alpha)$. Set $\sigma = 0$, $b = 0$ and $\nu = \sum_{n \in \mathbb{N}} a_n^{-\alpha} (c_+ \delta_{a_n} + c_- \delta_{-a_n})$ for $a_n = 2^{-\eta^n}$.
- * Then X satisfies $\beta_- = 0 < \alpha = \beta_+ \in (1, 2)$.

Orey's process

The following Lévy process satisfies $\liminf_{u \rightarrow \infty} |\psi(u)/u| < \infty$ and $\limsup_{u \rightarrow \infty} |\psi(u)/u| = \infty$, making Corollary 2 inapplicable.

Example 9.2

- * Fix any $\alpha \in (1, 2)$, $c_{\pm} \geq 0$ with $c_+ + c_- > 0$ and integer $\eta > 2/(2 - \alpha)$. Set $\sigma = 0$, $b = 0$ and $\nu = \sum_{n \in \mathbb{N}} a_n^{-\alpha} (c_+ \delta_{a_n} + c_- \delta_{-a_n})$ for $a_n = 2^{-\eta^n}$.
- * Then X satisfies $\beta_- = 0 < \alpha = \beta_+ \in (1, 2)$.
- * If $c_+ \neq c_-$, then X is A by Proposition 6.

Orey's process

The following Lévy process satisfies $\liminf_{u \rightarrow \infty} |\psi(u)/u| < \infty$ and $\limsup_{u \rightarrow \infty} |\psi(u)/u| = \infty$, making Corollary 2 inapplicable.

Example 9.2

- * Fix any $\alpha \in (1, 2)$, $c_{\pm} \geq 0$ with $c_+ + c_- > 0$ and integer $\eta > 2/(2 - \alpha)$. Set $\sigma = 0$, $b = 0$ and $\nu = \sum_{n \in \mathbb{N}} a_n^{-\alpha} (c_+ \delta_{a_n} + c_- \delta_{-a_n})$ for $a_n = 2^{-\eta^n}$.
- * Then X satisfies $\beta_- = 0 < \alpha = \beta_+ \in (1, 2)$.
- * If $c_+ \neq c_-$, then X is A by Proposition 6.
- * If $c_+ = c_-$, then $\mathfrak{s}_1(0) = \infty$, suggesting X may be SE (open problem!).

Sums of A and SE

Any of the following are possible for independent summands:

- * $A + A = A,$

- * $A + SE = A,$

- * $SE + SE = SE,$

- * $A + A = SE,$

- * $A + SE = SE,$

- * $SE + SE = A.$

Sums of A and SE

Any of the following are possible for independent summands:

$$* A + A = A,$$

$$* A + SE = A,$$

$$* SE + SE = SE,$$

$$* A + A = SE,$$

$$* A + SE = SE,$$

$$* SE + SE = A.$$

These examples can be generated with absolutely continuous Lévy measures with regularly varying densities (proofs are based on elementary estimates and Karamata's theorem):

$$\frac{\nu(dx)}{dx} = |x|^{-2}(\varphi_+(x)\mathbb{1}_{(0,1)}(x) + \varphi_-(x)\mathbb{1}_{(-1,0)}(x)), \quad x \in \mathbb{R}.$$

Sums of A and SE

Any of the following are possible for independent summands:

$$* A + A = A,$$

$$* A + SE = A,$$

$$* SE + SE = SE,$$

$$* A + A = SE,$$

$$* A + SE = SE,$$

$$* SE + SE = A.$$

These examples can be generated with absolutely continuous Lévy measures with regularly varying densities (proofs are based on elementary estimates and Karamata's theorem):

$$\frac{\nu(dx)}{dx} = |x|^{-2}(\varphi_+(x)\mathbb{1}_{(0,1)}(x) + \varphi_-(x)\mathbb{1}_{(-1,0)}(x)), \quad x \in \mathbb{R}.$$

The proof requires analysing

$$\mathfrak{s}_1(r) = \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du$$

Sums of A and SE

Any of the following are possible for independent summands:

$$* A + A = A,$$

$$* A + SE = A,$$

$$* SE + SE = SE,$$

$$* A + A = SE,$$

$$* A + SE = SE,$$

$$* SE + SE = A.$$

These examples can be generated with absolutely continuous Lévy measures with regularly varying densities (proofs are based on elementary estimates and Karamata's theorem):

$$\frac{\nu(dx)}{dx} = |x|^{-2}(\varphi_+(x)\mathbb{1}_{(0,1)}(x) + \varphi_-(x)\mathbb{1}_{(-1,0)}(x)), \quad x \in \mathbb{R}.$$

The proof requires analysing

$$\mathfrak{s}_1(r) = \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \quad \text{balance of } \varphi_+ \text{ and } \varphi_-!$$

Sums of A and SE

Any of the following are possible for independent summands:

$$* A + A = A,$$

$$* A + SE = A,$$

$$* SE + SE = SE,$$

$$* A + A = SE,$$

$$* A + SE = SE,$$

$$* SE + SE = A.$$

These examples can be generated with absolutely continuous Lévy measures with regularly varying densities (proofs are based on elementary estimates and Karamata's theorem):

$$\frac{\nu(dx)}{dx} = |x|^{-2}(\varphi_+(x)\mathbb{1}_{(0,1)}(x) + \varphi_-(x)\mathbb{1}_{(-1,0)}(x)), \quad x \in \mathbb{R}.$$

The proof requires analysing

$$\mathfrak{s}_1(r) = \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \quad \text{balance of } \varphi_+ \text{ and } \varphi_-!$$

Sums of A and SE

Any of the following are possible for independent summands:

$$* A + A = A,$$

$$* A + SE = A,$$

$$* SE + SE = SE,$$

$$* A + A = SE,$$

$$* A + SE = SE,$$

$$* SE + SE = A.$$

These examples can be generated with absolutely continuous Lévy measures with regularly varying densities (proofs are based on elementary estimates and Karamata's theorem):

$$\frac{\nu(dx)}{dx} = |x|^{-2}(\wp_+(x)\mathbb{1}_{(0,1)}(x) + \wp_-(x)\mathbb{1}_{(-1,0)}(x)), \quad x \in \mathbb{R}.$$

The proof requires analysing

$$\mathfrak{s}_1(r) = \int_{\mathbb{R}} \Re \frac{1}{1 + iur - \psi(u)} du \quad \text{balance of } \wp_+ \text{ and } \wp_-!$$

Moreover, within this class we may *also* generate examples of SE processes with $\liminf_{|u| \rightarrow \infty} |\psi(u)/u| < \infty = \limsup_{|u| \rightarrow \infty} |\psi(u)/u|$.

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0.

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0. **Number of visits is not the right quantity to determine SE or A!**
- * The occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t) t^{-1} dt$, $I \subset \mathbb{R}$, has mean $\int_0^1 \mathbb{P}(X_t/t \in I) t^{-1} dt$ which, by Theorem 1, needs to be infinite for X to be SE.

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0. **Number of visits is not the right quantity to determine SE or A!**
- * The occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t) t^{-1} dt$, $I \subset \mathbb{R}$, has mean $\int_0^1 \mathbb{P}(X_t/t \in I) t^{-1} dt$ which, by Theorem 1, needs to be infinite for X to be SE.
- * Our 0–1 law then suggests that $\mathcal{T}(I)$ needs to be infinite (resp. finite) for all bounded intervals I a.s. for X to be SE (resp. A).

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0. **Number of visits is not the right quantity to determine SE or A!**
- * The occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t) t^{-1} dt$, $I \subset \mathbb{R}$, has mean $\int_0^1 \mathbb{P}(X_t/t \in I) t^{-1} dt$ which, by Theorem 1, needs to be infinite for X to be SE.
- * Our 0–1 law then suggests that $\mathcal{T}(I)$ needs to be infinite (resp. finite) for all bounded intervals I a.s. for X to be SE (resp. A). (**Open problem!**)

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0. **Number of visits is not the right quantity to determine SE or A!**
- * The occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t) t^{-1} dt$, $I \subset \mathbb{R}$, has mean $\int_0^1 \mathbb{P}(X_t/t \in I) t^{-1} dt$ which, by Theorem 1, needs to be infinite for X to be SE.
- * Our 0–1 law then suggests that $\mathcal{T}(I)$ needs to be infinite (resp. finite) for all bounded intervals I a.s. for X to be SE (resp. A). **(Open problem!)**

Having characterised the continuity of C' , it is also natural to enquire about the modulus of continuity of C' at such points.

Concluding remarks and further work

It appears that the proof of Conjecture 1 would require a better understanding of the occupation measure of X_t/t as $t \rightarrow 0$:

- * If X is abrupt with $s_1(r) < \infty$ for all $r \in \mathbb{R}$, then X_t/t visits all bounded intervals immediately after time 0. **Number of visits is not the right quantity to determine SE or A!**
- * The occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t) t^{-1} dt$, $I \subset \mathbb{R}$, has mean $\int_0^1 \mathbb{P}(X_t/t \in I) t^{-1} dt$ which, by Theorem 1, needs to be infinite for X to be SE.
- * Our 0–1 law then suggests that $\mathcal{T}(I)$ needs to be infinite (resp. finite) for all bounded intervals I a.s. for X to be SE (resp. A). **(Open problem!)**

Having characterised the continuity of C' , it is also natural to enquire about the modulus of continuity of C' at such points. **Upcoming work on this topic!**

- [1] Jean Bertoin. “Regularity of the half-line for Lévy processes”. *Bull. Sci. Math.* 121.5 (1997), pp. 345–354. ISSN: 0007-4497.
- [2] Jean Bertoin. “The convex minorant of the Cauchy process”. *Electron. Comm. Probab.* 5 (2000), pp. 51–55. ISSN: 1083-589X. DOI: 10.1214/ECP.v5-1017. URL: <https://doi.org/10.1214/ECP.v5-1017>.
- [3] Jorge Ignacio González Cázares and Aleksandar Mijatović. “Convex minorants and the fluctuation theory of Lévy processes”. (2021). arXiv: 2105.15060.
- [4] Vincent Vigon. “Abrupt Lévy processes”. *Stochastic Process. Appl.* 103.1 (2003), pp. 155–168. ISSN: 0304-4149. DOI: 10.1016/S0304-4149(02)00186-2. URL: [https://doi.org/10.1016/S0304-4149\(02\)00186-2](https://doi.org/10.1016/S0304-4149(02)00186-2).
- [5] Vincent Vigon. *Dérivées de Dini de processus de Lévy*. Retrieved 1st September, 2021, from <https://drive.google.com/file/d/0B5ZYbxojmGIadE9U0V9KR0QyT1E/view?usp=sharing&resourcekey=0-ZIypnqJtKmdvZoBpR0riTw>. 2016.

The arxiv link to the article is in the description of the video.

Thank you for watching.