

Convex minorants and the fluctuation theory of Lévy processes

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and

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May 2021

Research supported by **EPSRC** (EP/P003818/1, EP/V009478/1 & EP/N510129/1)

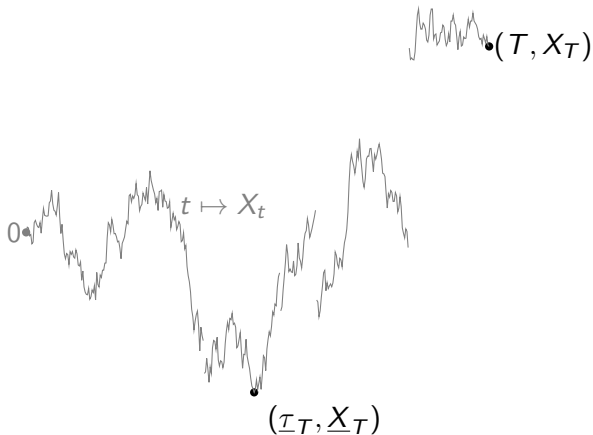
A simple approach to the fluctuations of Lévy process X



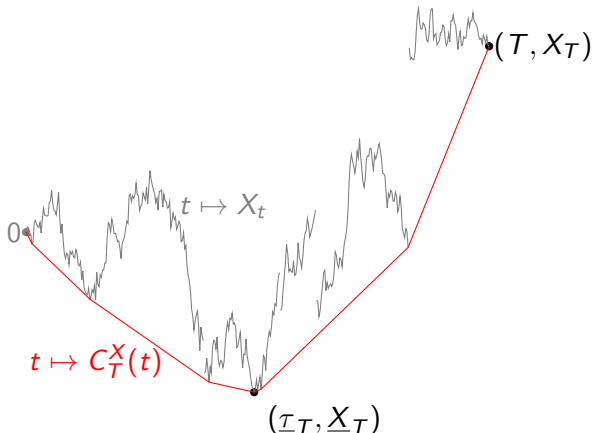
(T, X_T)



A simple approach to the fluctuations of Lévy process X

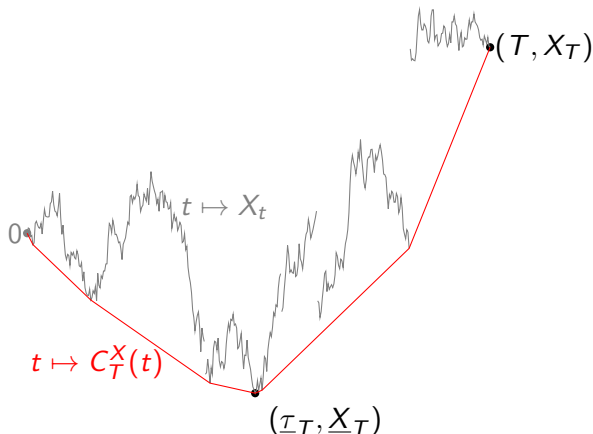


A simple approach to the fluctuations of Lévy process X



- (I) Characterise the law of convex minorant C_T^X of X as *piecewise linear (PwL) convex function*.

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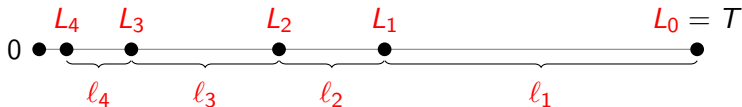


- (I) Characterise the law of convex minorant C_T^X of X as *piecewise linear (PwL) convex function*.
- (II) *Fluctuation theory* follows easily from (I).

(II) Extrema of a Lévy process X at a fixed time T

Theorem

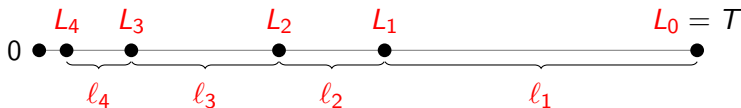
Let X be independent of stick-breaking process ℓ on $[0, T]$, ie for iid $U_n \sim U(0, 1)$, $L_0 = T$, $L_n = L_{n-1} U_n$, $\ell_n = L_{n-1} - L_n$ for $n \in \mathbb{N}$:



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Let $\underline{X}_T = \inf_{t \in [0, T]} X_t$ and $\underline{\tau}_T$ the time $\underline{X}_T$ is attained. Then

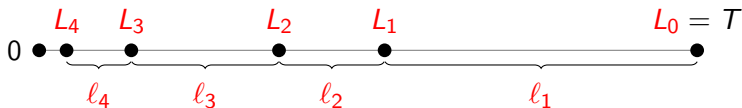
$$(\underline{X}_T, \underline{X}_T, \underline{\tau}_T) \stackrel{d}{=} \sum_{n=1}^{\infty} (\xi_n, \min\{\xi_n, 0\}, \ell_n \mathbb{1}_{\{\xi_n < 0\}}),$$

where, given ℓ , $\xi_n \sim F(\ell_n, \cdot)$ are independent for $n \in \mathbb{N}$ (here we denote $F(t, dx) = \mathbb{P}(X_t \in dx)$ for $(t, x) \in (0, \infty) \times \mathbb{R}$).

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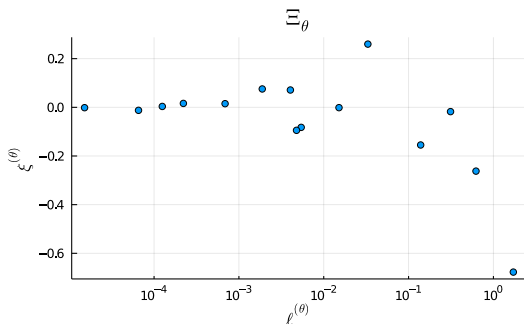
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Proof Direct corollary of characterisation in (I) two slides on ...

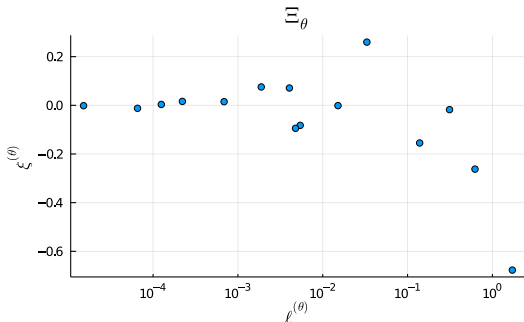
(II) Extrema of X at an independent exponential time T_θ

Let $\ell^{(\theta)} = (\ell_n^{(\theta)})_{n \in \mathbb{N}}$ a stick-breaking process on $[0, T_\theta]$. Given $\ell^{(\theta)}$, independent $\xi_n^{(\theta)}$ follow $F(\ell_n^{(\theta)}, \cdot)$. Then PPP $\Xi_\theta = \sum_{n=1}^{\infty} \delta_{(\ell_n^{(\theta)}, \xi_n^{(\theta)})}$ on $(0, \infty) \times \mathbb{R}$ has intensity $\mu_\theta(dt, dx) = t^{-1}e^{-\theta t} F(t, dx)dt$.



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Let $A_+ = [0, \infty)$, $A_- = (-\infty, 0)$ & $B_\pm = (0, \infty) \times A_\pm$. Then

$$(\underline{X}_{T_\theta}, \underline{X}_{T_\theta} - X_{T_\theta}) \stackrel{d}{=} \int_{(0, \infty) \times \mathbb{R}} (x \mathbb{1}_{B_-}, -x \mathbb{1}_{B_+}) \Xi_\theta(dt, dx) \quad (1)$$

holds by SB-representation at fixed time (condition on $T_\theta = T$).

(II) Wiener-Hopf factorization (and more ...)

Theorem

$\underline{X}_{T_\theta}$ and $X_{T_\theta} - \underline{X}_{T_\theta}$ are independent, i.d. and

$$\mathbb{E}[e^{u\underline{X}_{T_\theta}}] = \varphi_+(\theta, u), \quad \mathbb{E}[e^{u(\underline{X}_{T_\theta} - X_{T_\theta})}] = \varphi_-(\theta, u),$$

for any $u \in \mathbb{C}$ with $\Re u \geq 0$ and $\theta \in (0, \infty)$, where $\varphi_\pm$ is defined by

$$\varphi_\pm(\theta, u) := \exp \left(\int_0^\infty \int_{A_\pm} (e^{-u|x|} - 1) t^{-1} e^{-\theta t} \mathbb{P}(X_t \in dx) dt \right).$$

Proof Equality in law (1), $B_+ \cap B_- = \emptyset$ and Campbell's formula.

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Analogous simple arguments yield the full **WF factorisation**, **Rogozin's criterion** for regularity of zero, **Spitzer's formulas** for expected supremum etc, as well as **new** fluctuation identities, see paper for details.

(I) The law of the convex minorant C_T^X of X

Theorem (SB-representation theorem (SBRT))

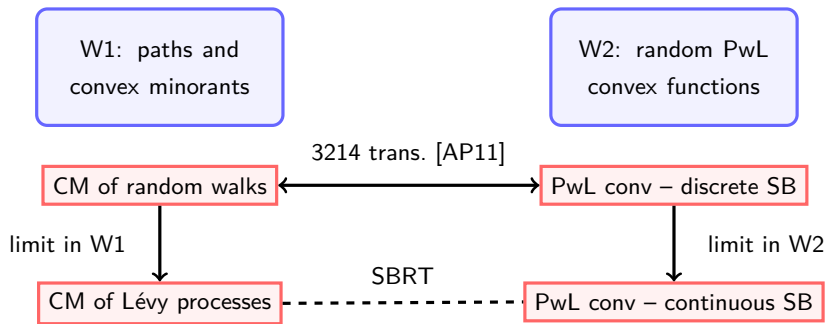
Let $(\ell_n)_{n \in \mathbb{N}}$ be a stick-breaking process on $[0, T]$ independent of Lévy process X . Then the convex minorant C_T^X of X has the same law as the PwL convex function on $[0, T]$ given by:

$$t \mapsto \sum_{n=1}^{\infty} \xi_n \min\{(t - a_n)^+ / \ell_n, 1\}, \quad \text{where } \xi_n := X_{L_{n-1}} - X_{L_n} \text{ and}$$

$$a_n := \sum_{k=1}^{\infty} \ell_k \cdot \mathbb{1}_{\{\xi_k / \ell_k < \xi_n / \ell_n\}} + \sum_{k=1}^{n-1} \ell_k \cdot \mathbb{1}_{\{\xi_k / \ell_k = \xi_n / \ell_n\}}, \quad n \in \mathbb{N}.$$

In particular, the face of the piecewise linear function with horizontal length ℓ_n has vertical height ξ_n .

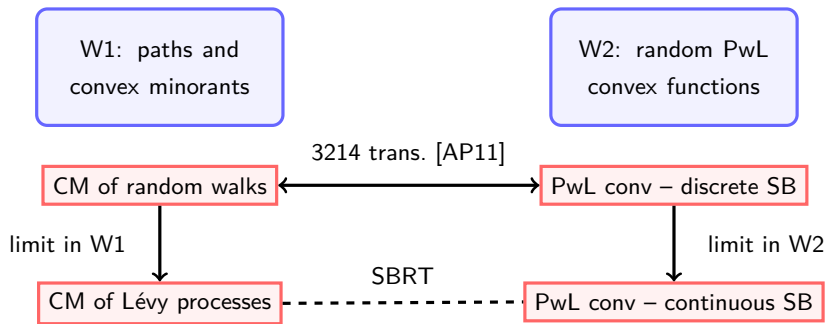
(I) Proof of SBRT: connecting worlds W1 & W2



Remarks

This elementary proof uses independence & stationarity of increments and the càdlàg property of paths **only**!

(I) Proof of SBRT: connecting worlds W1 & W2

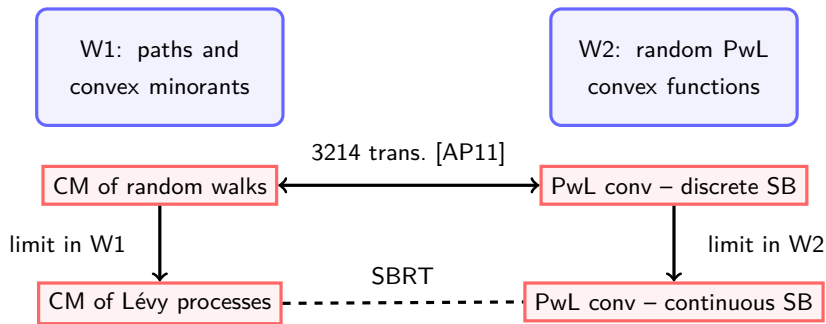


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[PUB12] proves SBRT by extending 3214 to diffuse Lévy processes

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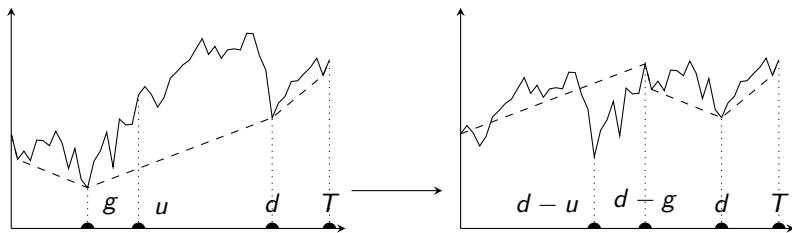
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[PUB12] proves SBRT by extending 3214 to diffuse Lévy processes

Proof in [PUB12] relies on fluctuation theory of X

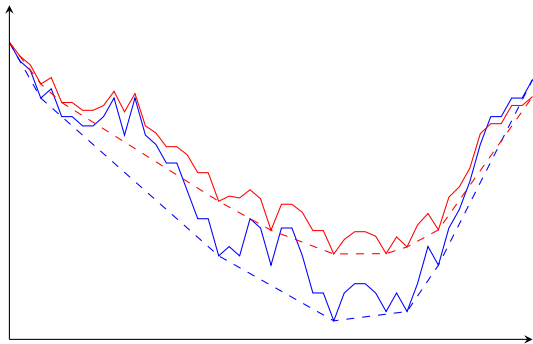
(I) 3214 transform [AP11]



Left: random walk (*solid*) and its convex minorant C_T (*dashed*)
($u \in (0, T)$) and endpoints $\{g, d\}$ of a face of C_T containing u)

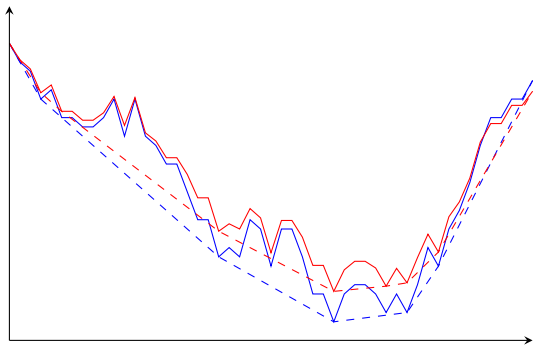
Right: 3214 transform

(I) W1 limit: convex minorants



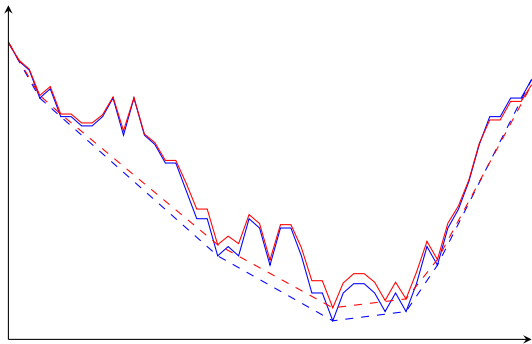
Convex minorants of nested skeletons of X converge to C_T^X as mesh goes to zero (only requires càdlàg paths)

(I) W1 limit: convex minorants



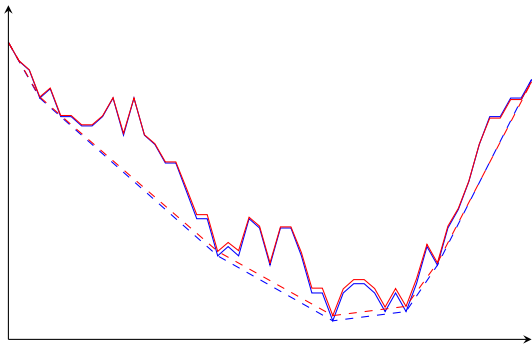
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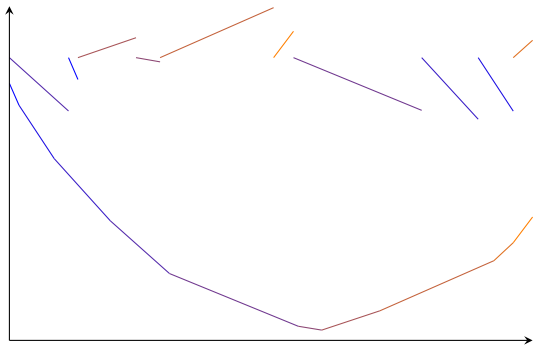
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(I) Piecewise linear convex functions



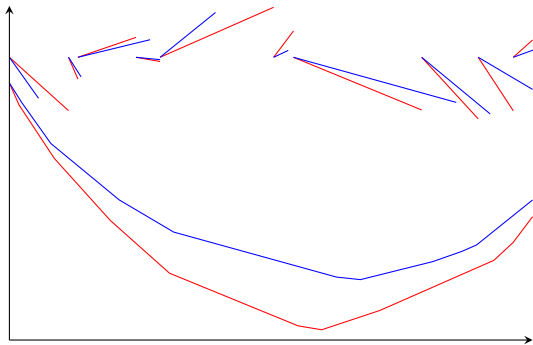
A unique convex function can be created given a sequence of segments (their lengths and heights)

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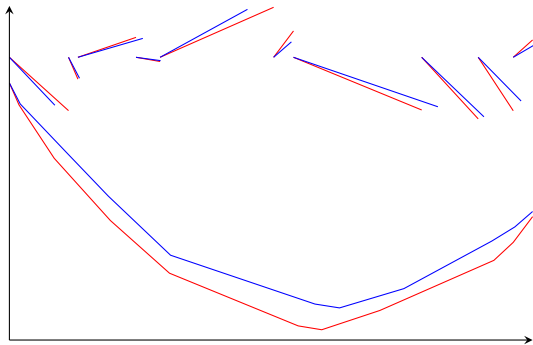
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(I) W2 limit: piecewise linear convex functions



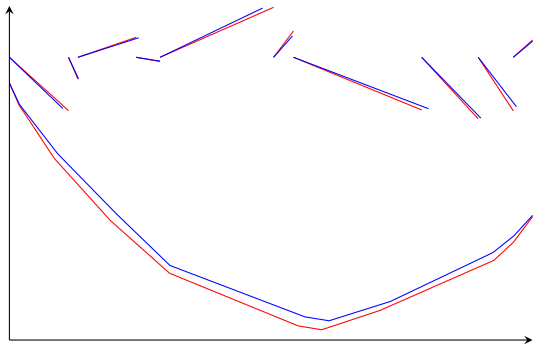
Convergent sequences of segments make the corresponding piecewise linear convex functions converge.

(I) W2 limit: piecewise linear convex functions



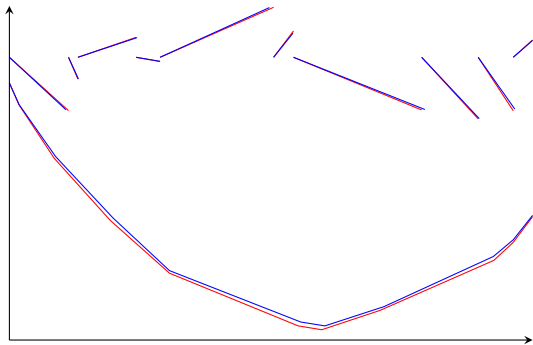
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References I

- [AP11] Josh Abramson and Jim Pitman, *Concave majorants of random walks and related Poisson processes*, *Combin. Probab. Comput.* **20** (2011), no. 5, 651–682. MR 2825583
- [PUB12] Jim Pitman and Gerónimo Uribe Bravo, *The convex minorant of a Lévy process*, *Ann. Probab.* **40** (2012), no. 4, 1636–1674. MR 2978134

arXiv link to the paper with the details of the proofs is in the description of the video.

Thank you for watching!