

Asymptotic shape of the concave majorant of a Lévy process

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June 2021


$$t \mapsto X_t$$
 $\bullet (T, X_T)$

Figure: A sample path of a Lévy process X on the interval $[0, T]$.

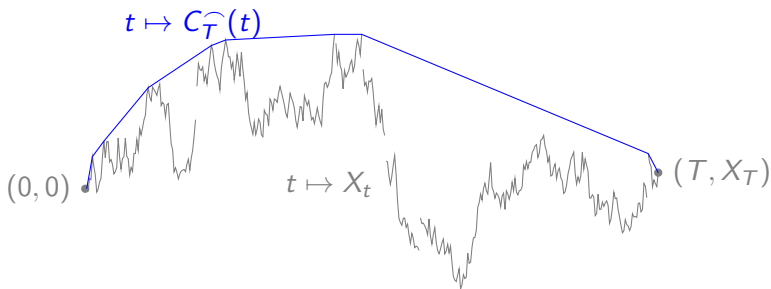


Figure: A sample path of a Lévy process X on the interval $[0, T]$ and the graph of the concave majorant $\widehat{C}_T$.

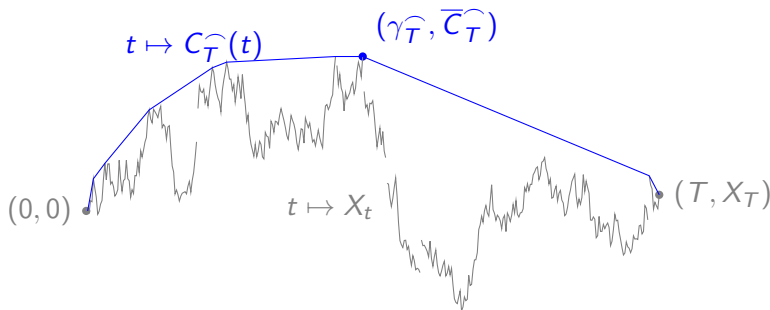
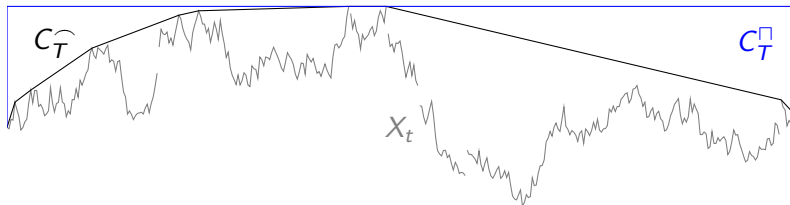


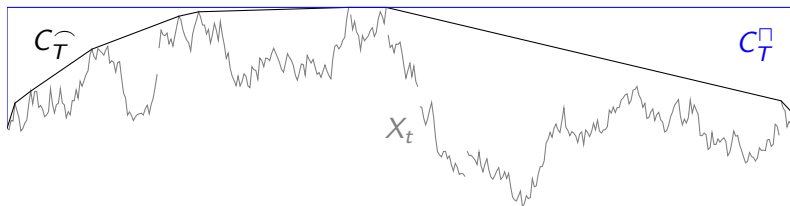
Figure: A sample path of a Lévy process X on the interval $[0, T]$, the graph of the concave majorant C_T and the time and space position of the supremum $(\gamma_T, \bar{C}_T)$.



- Let $\Upsilon_{\widehat{C}_T}$ denote the length of $\widehat{C}_T$ over $[0, T]$, then

$$1 \leq \Upsilon_{\widehat{C}_T}/T \leq (T + 2\overline{C}_T - \widehat{C}_T(T))/T.$$

- If $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[|X_1|^{1+\epsilon}] < \infty$ for some $\epsilon > 0$, then $\Upsilon_{\widehat{C}_T}/T \rightarrow 1$ a.s. as $T \rightarrow \infty$.



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- Interested in describing the fluctuations around the asymptotic mean of $\widehat{C}_T$.
- Distributional limit theorems for the centered statistics and limit dependence.

Theorem

Let $X = (X_t)_{t \geq 0}$ be a Lévy process with Lévy measure ν . Assume $\mathbb{E}[X_1] = 0$ and $\sigma := \sqrt{\mathbb{E}[X_1^2]} \in (0, \infty)$. For $T > 0$ define $\Theta(T) := \frac{1}{2} \int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\}) \nu(dx)$.

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$$\left(\frac{\Upsilon_T - T - (\sigma^2/2)H_T + \Theta(T)}{\sqrt{\log T}}, \frac{H_T - \log T}{\sqrt{\log T}}, \frac{\overline{C}_T}{\sqrt{T}}, \frac{C_T(T)}{\sqrt{T}}, \frac{\gamma_T}{T} \right) \xrightarrow{d} \left(\frac{\sigma^2}{\sqrt{2}} Z_1, Z_2, \sigma \overline{B}_1, \sigma B_1, \rho \right),$$

- standard BM $B = (B_t)_{t \geq 0}$, $\overline{B}_1 := \sup_{t \in [0,1]} B_t$ and $\rho \in [0, 1]$ a.s. unique time: $B_\rho = \overline{B}_1$.
- Standard normal random variables Z_1 and Z_2 are independent, independent of B .

Corollary

Under the assumptions of Theorem 1, we have

$$\frac{1}{\sqrt{\log T}} \left(\Upsilon_{\widehat{T}} - T - \frac{\sigma^2}{2} \log T + \Theta(T) \right) \xrightarrow{d} \frac{\sqrt{3}}{2} \sigma^2 Z, \quad \text{as } T \rightarrow \infty,$$

where:

- $\Theta(T) = o(\log T)$
- $Z \sim N(0, 1)$.

If $\int_{\mathbb{R}} x^2 \log^+(|x|)^{1/2} \nu(dx) < \infty$, then $\Theta(T) = o(\sqrt{\log T})$, and thus

$$\frac{1}{\sqrt{\log T}} \left(\Upsilon_{\widehat{T}} - T - \frac{\sigma^2}{2} \log T \right) \xrightarrow{d} \frac{\sqrt{3}}{2} \sigma^2 Z, \quad \text{as } T \rightarrow \infty.$$

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Domain of attraction

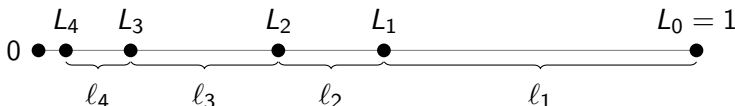
- Lévy process X is in the domain of attraction of an α -stable law for some $\alpha \in (0, 2]$
- Assume the existence of a positive function $(a_T)_{T \in \mathbb{R}_+}$, and an α -stable Lévy process $S_\alpha = (S_\alpha(t))_{t \in [0,1]}$ such that $X_T/a_T \xrightarrow{d} S_\alpha(1)$ as $T \rightarrow \infty$.

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- a_T is regular varying with index $\alpha \in (0, 2) \setminus \{1\}$. Therefore, a_T has the form $T^{1/\alpha} l(T)$, where l is a slowly varying function.

Stick-breaking process

- Let $(\ell_n)_{n \in \mathbb{N}}$ be a uniform stick-breaking process on $[0, 1]$ with stick remainders $(L_n)_{n \in \mathbb{N} \cup \{0\}}$ given by an iid- $U(0, 1)$ sequence $(V_n)_{n \in \mathbb{N}}$.
- $L_0 = 1$, $\ell_n = V_n L_{n-1}$ and $L_n = L_{n-1} - \ell_n$ for $n \in \mathbb{N}$.



Theorem

Assume X is in the domain of attraction of an α -stable law with $\alpha \in (1, 2)$ and $\mathbb{E}[X_1] = 0$. Then

$$\left(\frac{T}{a_T^2} (\Upsilon_T - T), \frac{\widehat{\overline{C}}_T}{a_T}, \frac{\widehat{C}_T(T)}{a_T}, \frac{\widehat{\gamma}_T}{T} \right) \xrightarrow{d} \sum_{n=1}^{\infty} \left(\frac{1}{2} \ell_n^{2/\alpha-1} (S_\alpha^{(n)})^2, \ell_n^{1/\alpha} (S_\alpha^{(n)})^+, \ell_n^{1/\alpha} S_\alpha^{(n)}, \ell_n \mathbb{1}_{(0,\infty)}(S_\alpha^{(n)}) \right),$$

as $T \rightarrow \infty$, where

- $(\ell_n)_{n \in \mathbb{N}}$ is a uniform stick-breaking process,
- $(S_\alpha^{(n)})_{n \in \mathbb{N}}$ sequence of independent copies of $S_\alpha(1)$,
- $(\ell_n)_{n \in \mathbb{N}}$ and $(S_\alpha^{(n)})_{n \in \mathbb{N}}$ are independent.

Theorem

Assume X is in the domain of attraction of the α -stable law $S_\alpha(1)$ for $\alpha \in (0, 1)$. Define

$$\Lambda_T^1 := \left(\frac{\Upsilon_T^\wedge}{a_T}, \frac{\overline{C}_T^\wedge}{a_T}, \frac{C_T^\wedge(T)}{a_T}, \frac{\gamma_T^\wedge}{T} \right),$$

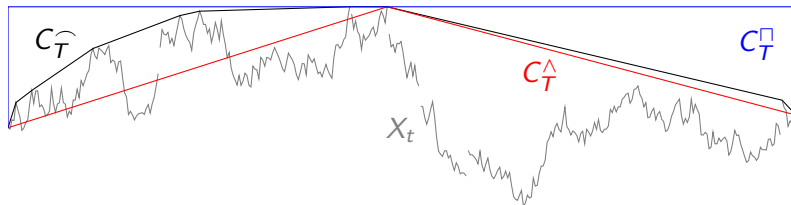
$$\Lambda^1 := (2\overline{S}_\alpha(1) - S_\alpha(1), \overline{S}_\alpha(1), S_\alpha(1), \gamma^{\alpha\smile}),$$

$$\Lambda_T^2 := \left(\frac{\Upsilon_T^\smile}{a_T}, \frac{\underline{C}_T^\smile}{a_T}, \frac{C_T^\smile(T)}{a_T}, \frac{\gamma_T^\smile}{T} \right),$$

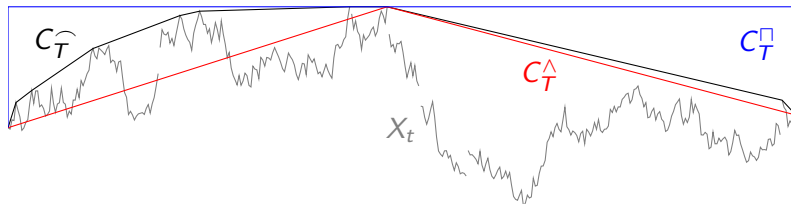
$$\Lambda^2 := (S_\alpha(1) - 2\underline{S}_\alpha(t), \underline{S}_\alpha(t), S_\alpha(1), \gamma^{\alpha\smile}).$$

Then $(\Lambda_T^1, \Lambda_T^2) \xrightarrow{d} (\Lambda^1, \Lambda^2)$ jointly as $T \rightarrow \infty$.

Overview



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Setting	Scaling of X_T	$\Upsilon_T^{\hat{\alpha}}$	$\Upsilon_T^{\square}$	$\Upsilon_T^{\square}$
$\alpha = 2$	$a_T = \sqrt{T}$	1	$\sqrt{\log T}$	$\sqrt{T}$
$1 < \alpha < 2$	$a_T = T^{1/\alpha}l(T)$	$T^{2/\alpha-1}l(T)^2$	$T^{2/\alpha-1}l(T)^2$	$T^{1/\alpha}l(T)$
$0 < \alpha < 1$	$a_T = T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$

Table: The table shows the scaling functions (after centering) in the weak limits of the lengths $\Upsilon_T^{\hat{\alpha}}$, $\Upsilon_T^{\square}$ and $\Upsilon_T^{\square}$ under the assumptions of the corresponding theorems with a_T as in definition of domain of attraction.

Main idea is [2, Thm 12], which for any Lévy process and time horizon $T > 0$ imply:

$$(\Upsilon_T, H'_T, C_T(T), \overline{C}_T, \gamma_T) \\ \stackrel{d}{=} \sum_{n=1}^{\infty} \left(\sqrt{(T\ell_n)^2 + \xi_n^2}, \mathbb{1}_{\{T\ell_n \geq 1\}}, \xi_n, \xi_n^+, T\ell_n \mathbb{1}_{\{\xi_n > 0\}} \right),$$

- $\xi_n := X_{TL_{n-1}} - X_{TL_n}$
- H'_T is a random variable such that $|H_T - H'_T|$ is bounded in L^1 as $T \rightarrow \infty$
- ℓ is a uniform stick-breaking process independent of X .

A Gaussian approximation theorem for Lévy processes [1]

Theorem

Let X be a Lévy process with $\mathbb{E}X_1 = 0$ and $\sigma^2 = \mathbb{E}X_1^2 \in (0, \infty)$. If $\nu \neq 0$, choose $\kappa \geq 1$ to satisfy $\nu((-\kappa, \kappa)) \in (0, \infty]$ and otherwise set $\kappa := 1$. Define:

- $\sigma_t^2 := \sigma^2 - \int_{\mathbb{R} \setminus (-\kappa\sqrt{t}, \kappa\sqrt{t})} x^2 \nu(dx)$
- $K(t) := \sup_{x \in \mathbb{R}} |\mathbb{P}(X_t/\sqrt{t} \leq x) - \Phi(x/\sigma_t)|$, for any $t > 0$
- Φ denotes the CDF of the standard normal random variable.

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Then

$$\int_1^\infty \frac{K(t)}{t} dt < \infty.$$



David Bang, Jorge Ignacio González Cázares, and Aleksandar Mijatović. *A Gaussian approximation theorem for Lévy processes*. 2021. arXiv: 2104.13855 [math.PR].



Jorge Ignacio González Cázares and Aleksandar Mijatović. “Convex minorants and the fluctuation theory of Lévy processes”. (2021). Proof explained in [YouTube video](#). arXiv: 2105.15060 [math.PR].

arXiv link to the paper “Asymptotic shape of the concave majorant of a Lévy process” on which this presentation is based is available in the description of this video below.

Thank you for watching!