

Monte Carlo algorithm for the extrema of tempered stable processes

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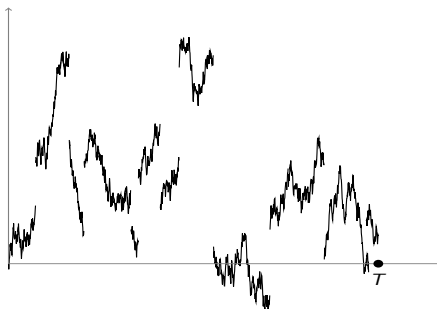


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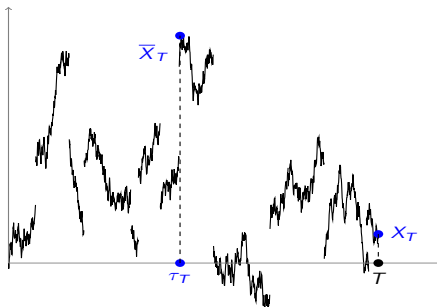
Object of study

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Objective: Monte Carlo estimation for functions $g(\chi)$, where $\chi = (X_T, \bar{X}_T, \tau_T)$, $\bar{X}_T = \sup_{t \in [0, T]} X_t$ is the path supremum and τ_T is its time location.

Motivation: mathematical finance

Under the exponential tempered stable model, the stock price $S = (S_t)_{t \geq 0}$ is given by $S_t = S_0 \exp(X_t)$. We have options whose payoffs depend on the path supremum $\bar{S}_T = S_0 \exp(\bar{X}_T)$.

- ▶ Hindsight call option with payoff $(\bar{S}_T - K)^+$
- ▶ Barrier type options with payoff $h(S_T)1\{\bar{S}_T > M\}$

Drawdown-based risk metrics:

- ▶ Mean drawdown $\mathbb{E}[\bar{S}_T - S_T]$
- ▶ Mean drawdown duration $\mathbb{E}[T - \tau_T]$
- ▶ The ulcer index $100\sqrt{\mathbb{E}[(S_T/\bar{S}_T - 1)^2]}$

Setting

Let $\lambda = (\lambda_+, \lambda_-) \in \mathbb{R}_+^2$ be different from the origin $\mathbf{0} = (0, 0)$.
The Lévy triplet $(0, \nu_\lambda, b_\lambda)$ is given by

$$\begin{aligned}\nu_\lambda(dx) &= e^{-\lambda_{\text{sgn}(x)}|x|} \nu(dx), \\ \frac{\nu(dx)}{dx} &= c_+ x^{-\alpha_+ - 1} \mathbb{1}_{x>0} + c_- |x|^{-\alpha_- - 1} \mathbb{1}_{x<0} \\ b_\lambda &= b + \int_{(-1,1)} x(e^{-\lambda_{\text{sgn}(x)}|x|} - 1) \nu(dx),\end{aligned}$$

where $b \in \mathbb{R}$, $c_\pm \geq 0$ and $\alpha_\pm \in [0, 2)$. We stress the dependence in λ , since $\lambda = \mathbf{0}$ reduces the model to a sum of stable processes.

Lévy process

Large jumps

This model allows:

- ▶ Polynomially arbitrary degree of jump activity (primarily determined by the small-jumps): $\alpha_{\pm} \in [0, 2)$ and
- ▶ Exponentially arbitrary tail behaviour: $\lambda \in \mathbb{R}_+^2$.

This is why many subclasses are widely used, including the popular CGMY process.

The stick-breaking representation

There exists a stick-breaking process ℓ_n (i.e. there are iid $U_n \sim U(0, 1)$ such that $L_0 = T$, $\ell_n = U_n L_{n-1}$, and $L_n = L_{n-1} - \ell_n$) such that for any $n \in \mathbb{N}$ there exists $Y \perp\!\!\!\perp \ell$ with $Y \stackrel{d}{=} X$ and

$$\chi = \begin{pmatrix} X_T \\ \bar{X}_T \\ \tau_T \end{pmatrix} = \sum_{m=1}^n \begin{pmatrix} Y_{L_{m-1}} - Y_{L_m} \\ (Y_{L_{m-1}} - Y_{L_m})^+ \\ \ell_m 1_{Y_{L_{m-1}} - Y_{L_m} > 0} \end{pmatrix} + \begin{pmatrix} Y_{L_n} \\ \bar{Y}_{L_n} \\ \tau_{L_n}(Y) \end{pmatrix}.$$

The SB approximation is

$$\chi = \begin{pmatrix} X_T \\ \bar{X}_T \\ \tau_T \end{pmatrix} = \sum_{m=1}^n \begin{pmatrix} Y_{L_{m-1}} - Y_{L_m} \\ (Y_{L_{m-1}} - Y_{L_m})^+ \\ \ell_m 1_{Y_{L_{m-1}} - Y_{L_m} > 0} \end{pmatrix} + \begin{pmatrix} Y_{L_n} \\ Y_{L_n}^+ \\ L_n 1_{Y_{L_n} > 0} \end{pmatrix}.$$

The error

The error of the SB approximation is geometrically small [GCMUB21], of order $\mathcal{O}(\eta^{-n})$ for $\eta = 1 + \min\{1, 1/\alpha\} \in [3/2, 2]$.

The problem: we generally cannot sample the increments exactly!

The solution: change of measure!

The downside: variance inflation.

The variance inflation is related to the coefficient

$$\mu_\lambda := \int_{\mathbb{R}} (e^{-\lambda_{\text{sgn}(x)}|x|} - 1 + \lambda_{\text{sgn}(x)}|x| \cdot \mathbb{1}_{(-1,1)}(x)) \nu(dx),$$

in the change-of-measure formula.

TSB-Alg

Unbiased simulation of $g(\chi_n)$

Require: Parameters $\lambda \in \mathbb{R}_+^2 \setminus \{0\}$, b , $c_{\pm}$ and $\alpha \in (0, 1)$, time horizon $T > 0$, test function g , approximation level $n \in \mathbb{N}$

- 1: Set $L_0 := T$ and compute μ_{λ}
- 2: **for** $k = 1, \dots, n$ **do**
- 3: Sample $U_k \sim U(0, 1)$ and put $L_k := U_k L_{k-1}$ and $\ell_k := L_{k-1} - L_k$
- 4: Sample $\xi_k^{(\pm)} \sim F^{(\pm)}(\ell_k, \cdot)$ and put $\xi_k := \xi_k^{(+)} + \xi_k^{(-)} + \ell_k b$
- 5: **end for**
- 6: Sample $\zeta_n^{(\pm)} \sim F^{(\pm)}(L_n, \cdot)$ and put $\zeta_n := \zeta_n^{(+)} + \zeta_n^{(-)} + L_n b$
- 7: Set $\chi_n := \sum_{k=1}^n (\xi_k, \max\{\xi_k, 0\}, \ell_k \mathbb{1}_{\{\xi_k > 0\}}) + (\zeta_n, \max\{\zeta_n, 0\}, L_n \mathbb{1}_{\{\zeta_n > 0\}})$
- 8: Set $Y_T^{(\pm)} := \zeta_n^{(\pm)} + \sum_{k=1}^n \xi_k^{(\pm)}$
- 9: **return** $g(\chi_n) \exp(-\lambda_+ Y_T^{(+)} + \lambda_- Y_T^{(-)} - \mu_{\lambda} T)$

where $F^{(+)}(t, \cdot)$ and $F^{(-)}(t, \cdot)$ are the laws of some (explicitly known) spectrally positive (resp. negative) α -stable process.

Confidence Intervals (CIs) & Computational Cost

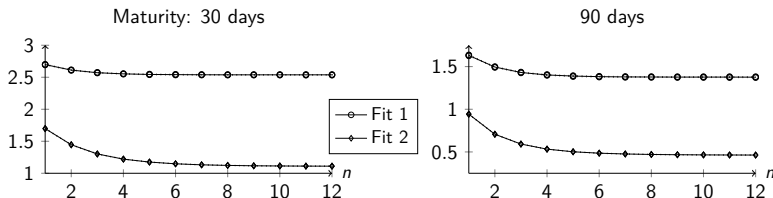
$$\begin{aligned} & |\mathbb{E}g(\chi) - \frac{1}{N} \sum_{i=1}^N g(\chi_n^i)| \\ & \leq \underbrace{|\mathbb{E}g(\chi) - \mathbb{E}g(\chi_n)|}_{B(n)} + \underbrace{|\mathbb{E}g(\chi_n) - \frac{1}{N} \sum_{i=1}^N g(\chi_n^i)|}_{MC_n(N)} \end{aligned}$$

For the SB approximation of X and Lipschitz g :

- ▶ Bias decays geometrically $B(n) = \mathcal{O}(\eta^n)$ for $\eta = 1 + \min\{1, 1/\alpha\} \in [3/2, 2]$
- ▶ (For fixed n) Monte-Carlo decay: $MC_n(N) = \mathcal{O}(N^{-1/2})$
- ▶ Computational cost of precision $\epsilon > 0$ is

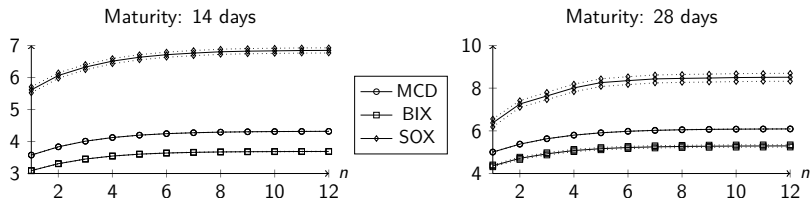
$$n \times N \propto |\log(\epsilon)| \epsilon^{-2}$$

Simulations



Solid lines indicate the estimated value of $\mathbb{E}g(\chi_n)$ for the USD/JPY currency pair and the function $g(\chi) = \max\{S_T - K, 0\} \mathbb{1}\{\bar{S}_T \leq M\}$ using $N = 10^6$ samples with $T \in \{\frac{30}{365}, \frac{90}{365}\}$, $K = 95$ and $M = 102$. *Dotted* lines (the symmetric bands around each solid line) indicate the confidence interval of the estimates with confidence level 95%. These confidence intervals are very tight, making them hard to discern in the plot.

Simulations



Solid lines indicate the estimated value of the ulcer index (UI) $100\sqrt{\mathbb{E}g(\chi_n)}$ using $N = 10^7$ samples, where $g(\chi) = (S_T/\bar{S}_T - 1)^2$. *Dotted* lines (the bands around each solid line) indicate the confidence interval of the estimates with confidence level 95%.

References I

[GCMUB21] Jorge I. González Cázares, Aleksandar Mijatović, and Gerónimo Uribe Bravo, *Geometrically convergent simulation of the extrema of Lévy processes*, to appear in Math. Oper. Res. (2021).

THANK YOU FOR YOUR ATTENTION

Paper available on the [arXiv](#) (see also the video description)