

Geometrically convergent simulation of the extrema of Lévy processes via **SB Algorithm**

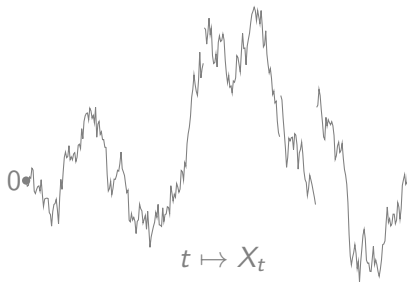
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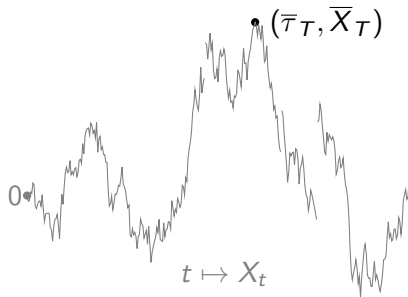
June 2021

Law of $(X_T, \bar{X}_T, \bar{\tau}_T)$ for *any* Lévy process X

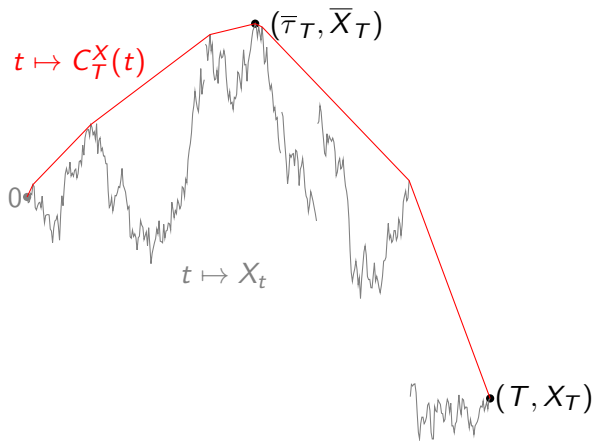


 (T, X_T)

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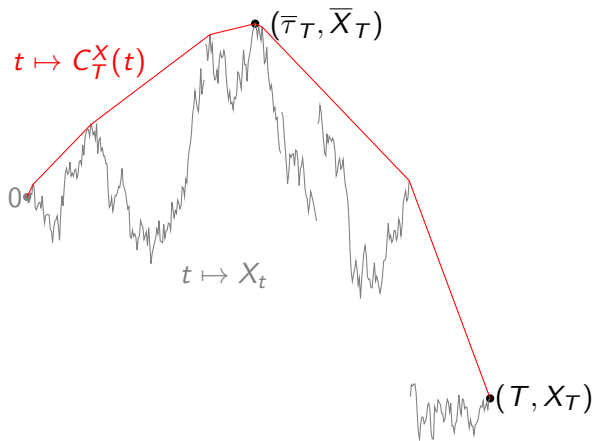


The concave majorant of *any* Lévy process X [CM21b]



We use a characterisation [CM21b] of the *law* of concave majorant C_T^X of X as a *piecewise linear concave* function.

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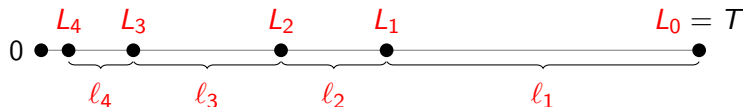
We use a characterisation [CM21b] of the *law* of concave majorant C_T^X of X as a *piecewise linear concave* function.

[CM21b] available at [arXiv](#) and explained in [video](#) on this channel.

Extrema of a Lévy process X at a fixed time T

Theorem

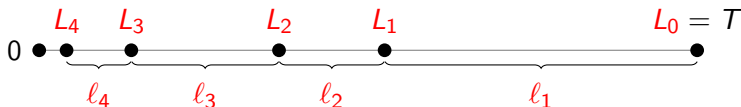
Let X be independent of stick-breaking process ℓ on $[0, T]$, ie for iid $U_n \sim U(0, 1)$, $L_0 = T$, $L_n = L_{n-1}U_n$, $\ell_n = L_{n-1} - L_n$ for $n \in \mathbb{N}$:



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Let $\bar{X}_T = \sup_{t \in [0, T]} X_t$ and $\bar{\tau}_T$ the time $\bar{X}_T$ is attained. Then

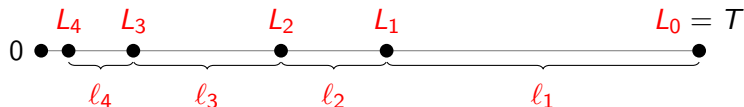
$$(X_T, \bar{X}_T, \bar{\tau}_T) \stackrel{d}{=} \sum_{n=1}^{\infty} (\xi_n, \max\{\xi_n, 0\}, \ell_n \mathbb{1}_{\{\xi_n > 0\}}),$$

where, given ℓ , $\xi_n = X_{L_{n-1}} - X_{L_n} \sim F(\ell_n, \cdot)$ are independent for $n \in \mathbb{N}$ (here we denote $F(t, dx) = \mathbb{P}(X_t \in dx)$ for $(t, x) \in (0, \infty) \times \mathbb{R}$).

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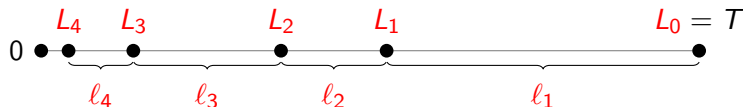
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SB-Algorithm (SB-Alg)

Stick-breaking approx. of $\chi := \sum_{n=1}^{\infty} (\xi_n, \max\{\xi_n, 0\}, \ell_n \mathbb{1}_{\{\xi_n > 0\}})$ is defined by

$$\chi_n := (X_{L_n}, X_{L_n}^+, L_n \mathbb{1}_{\{X_{L_n} > 0\}}) + \sum_{k=1}^n (\xi_k, \max\{\xi_k, 0\}, \ell_k \mathbb{1}_{\{\xi_k > 0\}}).$$

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*Simulation of the stick-breaking approximation via **SB-Alg**:*

Require: Time horizon $T > 0$ and number of steps $n \in \mathbb{N}$

- 1: Set $\Lambda_0 = T$, $\mathcal{X}_0 = (0, 0, 0)$
- 2: **for** $k = 1, \dots, n$ **do**
- 3: Sample $v_k \sim \text{U}(0, 1)$, put $\lambda_k = v_k \Lambda_{k-1}$ and $\Lambda_k = \Lambda_{k-1} - \lambda_k$
- 4: Sample $\xi_k \sim F(\lambda_k, \cdot)$, put $\mathcal{X}_k = \mathcal{X}_{k-1} + (\xi_k, \xi_k^+, \lambda_k \mathbb{1}_{\{\xi_k > 0\}})$
- 5: Sample $\varsigma_n \sim F(\Lambda_n, \cdot)$ and
- 6: **return** $\mathcal{X}_n + (\varsigma_n, \varsigma_n^+, \Lambda_n \mathbb{1}_{\{\varsigma_n > 0\}})$

Properties of SB-Alg

The recursive structure of the SB-process yields:

Theorem

The error $(0, \Delta_n, \delta_n) := \chi - \chi_n$ of χ_n satisfies
 $(\Delta_n, \delta_n) \stackrel{d}{=} (\bar{X}_{L_n} - X_{L_n}, \bar{\tau}_{L_n}(X) - L_n \mathbb{1}_{\{X_{L_n} > 0\}}).$

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- ▶ We have $|\delta_n| \leq L_n$ with $\mathbb{E}[(L_n)^p] \leq (1+p)^{-n}$ for $p > 0$
- ▶ $\mathbb{E}[|\Delta_n|^p] = \mathcal{O}(\eta_p^n)$ for some $\eta_p \in [1/2, 2/3]$ for every $p \geq 1$
(η_p depends on the Blumenthal-Gettoor index)
- ▶ If X is in the domain of attraction of a stable process:
 $X_t/a_t \xrightarrow{d} S$ as $t \rightarrow 0$, then the pair $(\Delta_n/a(L_n), \delta_n/L_n)$
converges weakly as $n \rightarrow \infty$
- ▶ The bias and level variances (for MLMC) can be controlled
and converge geometrically fast to 0

Monte Carlo complexity of **SB-Alg**

The complexity of Monte Carlo estimators of certain functions of interest of $(X_T, \bar{X}_T, \bar{\tau}_T)$ is of order $\mathcal{O}(\epsilon^{-2} \log \epsilon)$.

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These complexities apply to the following functions:

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- ▶ Discontinuous functions (including barrier options)

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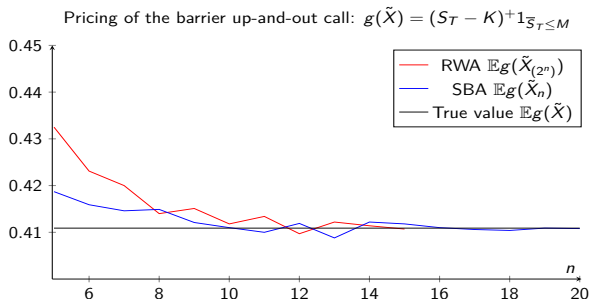
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We have a *CLT*, so we can construct asymptotic and non-asymptotic confidence intervals!

Numerical comparison: **SB-Alg** vs RW approximation

Let X have characteristics (b, σ^2, ν) with $\nu(x) \sim c|x|^{-2\alpha}$ as $x \rightarrow 0$ and $\nu(x) = \mathcal{O}(e^{-\sqrt{2\lambda}|x|})$ as $|x| \rightarrow \infty$.



Parameters: $(\alpha, \gamma, \lambda, \sigma, b, S_0, K, M) = (.75, .1, 4, .05, -.05, 2, 3, 5)$

References I

- [CM20] Jorge González Cázares and Aleksandar Mijatović, *Simulation of the drawdown and its duration in Lévy models via stick-breaking gaussian approximation*, available on [arXiv](#) and explained in short [video](#).
- [CM21a] ———, *Monte carlo algorithm for the extrema of tempered stable processes*, available on [arXiv](#) and explained in short [video](#).
- [CM21b] ———, *Convex minorants and the fluctuation theory of Lévy processes*, available on [arXiv](#), see also [video](#).
- [GCMUB21] Jorge I. González Cázares, Aleksandar Mijatović, and Gerónimo Uribe Bravo, *Geometrically convergent simulation of the extrema of Lévy processes*, To appear in Mathematics of Operations Research (2021), available on [arXiv](#).

Thank you for watching!