

Simulation of the position, supremum and the time of supremum of a Lévy process via stick-breaking Gaussian approximation

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and

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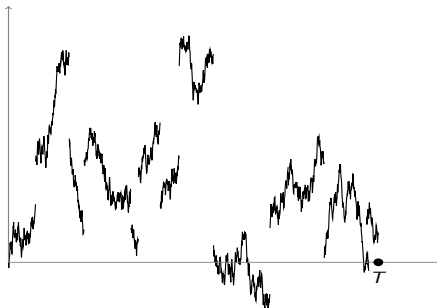


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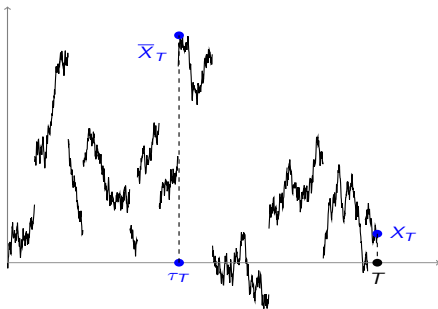
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$X = (X_t)_{t \in [0, T]}$ general Lévy process on $[0, T]$ for fixed $T > 0$.



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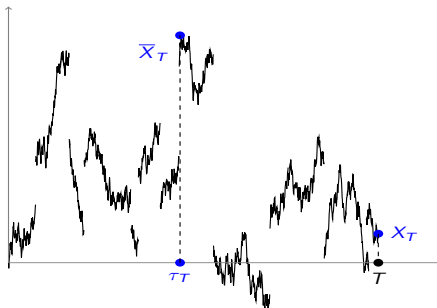
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Problem: Simulate $(X_T, \bar{X}_T, \tau_T)$ where $\bar{X}_T = \sup_{t \in [0, T]} X_t$ is its path supremum and $\tau_T = \tau_T(X)$ is its time location.

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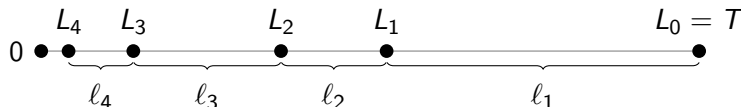


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Motivation: broad in applied probability, e.g. *drawdown* $\bar{S}_T - S_T$ and its *duration* $T - \tau_T(S)$ in exponential Lévy models $S = S_0 e^X$.

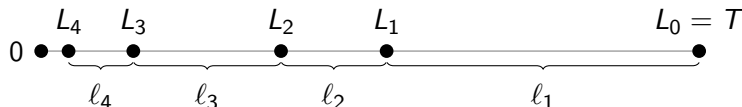
Stick-breaking approximation (SBA)

Stick-breaking process ℓ : $U_n \sim U(0, 1)$ iid, $L_0 = T$, $\ell_n = U_n L_{n-1}$, and $L_n = L_{n-1} - \ell_n$. Note: $\mathbb{E}[L_n] = T/2^n$.



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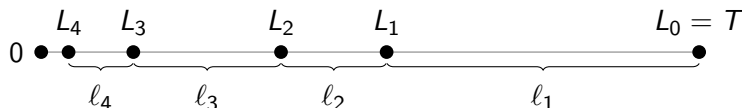


[GCMUB19, Thm 1]: $\exists$ coupling (X, Y) , independent of ℓ , such that $Y \stackrel{d}{=} X$ and

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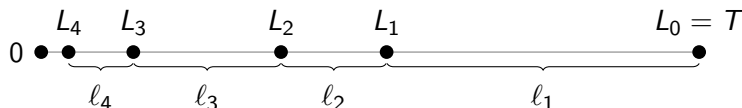
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requires the **ability** to **sample increment** X_t (for any $t > 0$) of X

Why is sampling X_t impossible in general?

Denote by (b, ν, σ^2) the triplet of the Lévy process X

Lévy-Itô decomposition of $\mathbf{X}$ at **level** $\kappa \in (0, 1]$:

$$X_t = b_\kappa t + \sigma B_t + \textcolor{red}{J}_t^{1,\kappa} + \textcolor{blue}{J}_t^{2,\kappa} \quad \text{for all } t > 0, \text{ where}$$

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- $b_\kappa = b - \int_{(-1,1) \setminus (-\kappa,\kappa)} x \nu(dx)$ & B standard BM
- $J^{1,\kappa}$ Lévy martingale with triplet $(0, \nu|_{(-\kappa,\kappa)}, 0)$
- $J^{2,\kappa}$ compound Poisson process with Lévy measure $\nu|_{\mathbb{R} \setminus (-\kappa,\kappa)}$
- all processes $(B, J^{1,\kappa}, J^{2,\kappa})$ independent

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Law of $J_t^{1,\kappa}$ is in general **intractable** for simulation purposes!

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We assume that:

- For any $\kappa \in (0, 1]$ we can compute

$$\bar{\nu}(\kappa) = \nu(\mathbb{R} \setminus (-\kappa, \kappa)) \quad \text{and} \quad \bar{\sigma}_\kappa^2 = \int_{(-\kappa, \kappa)} x^2 \nu(dx) = \mathbb{E}[(J_1^{1,\kappa})^2].$$

- We can sample from the distribution $\nu|_{\mathbb{R} \setminus (-\kappa, \kappa)} / \bar{\nu}(\kappa)$.

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The Gaussian approximation at level κ is given by

$$X_t^{(\kappa)} = b_\kappa t + \sigma B_t + \bar{\sigma}_\kappa W_t + J_t^{2,\kappa},$$

where $(W, J^{1,\kappa})$ is independent of B and $J^{2,\kappa}$.

How **not** to sample $(X_T^{(\kappa)}, \overline{X}_T^{(\kappa)}, \tau_T(X^{(\kappa)}))$

Denote $\tau_T^{(\kappa)} = \tau_T(X^{(\kappa)})$. Sample $(X_T^{(\kappa)}, \overline{X}_T^{(\kappa)}, \tau_T^{(\kappa)})$ as follows:

- Sample jump times $T_1, \dots, T_N$ of $J^{2,\kappa}$ on $[0, T]$; sample jump sizes $S_1, \dots, S_N$ as iid variables with law $\nu|_{\mathbb{R} \setminus (-\kappa, \kappa)} / \overline{\nu}(\kappa)$.
- Simulate the increment, maximum and time of the maximum of $\sigma B_t + \overline{\sigma}_\kappa W_t + b_\kappa t$ on each interval $[T_{k-1}, T_k)$, where $T_0 = 0$.
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This is **not a problem** for the simulation of the marginal $X_T^{(\kappa)}$.

Main questions

- Is it possible to reduce the sequential nature of the simulation of $(X_T^{(\kappa)}, \overline{X}_T^{(\kappa)}, \tau_T^{(\kappa)})$ and speed up its simulation?
- How close are the laws of the triplets $(X_T, \overline{X}_T, \tau_T(X))$ and $(X_T^{(\kappa)}, \overline{X}_T^{(\kappa)}, \tau_T^{(\kappa)})$ and how does this distance depend on $\kappa \in (0, 1]$ and $T > 0$?
- How do we implement a multilevel Monte Carlo (MLMC) scheme for this approximation?

Stick-breaking Gaussian (SBG) algorithm

SBA identity for $Y^{(\kappa)} \stackrel{d}{=} X^{(\kappa)}$ yields a fast & stable algorithm:

$$\begin{pmatrix} X_T^{(\kappa)} \\ \overline{X}_T^{(\kappa)} \\ \tau_T^{(\kappa)} \end{pmatrix} = \sum_{m=1}^n \begin{pmatrix} Y_{L_{m-1}^{(\kappa)}}^{(\kappa)} - Y_{L_m^{(\kappa)}}^{(\kappa)} \\ \left(Y_{L_{m-1}^{(\kappa)}}^{(\kappa)} - Y_{L_m^{(\kappa)}}^{(\kappa)} \right)^+ \\ \ell_m^{(\kappa)} \mathbb{1}_{\left\{ Y_{L_{m-1}^{(\kappa)}}^{(\kappa)} - Y_{L_m^{(\kappa)}}^{(\kappa)} > 0 \right\}} \end{pmatrix} + \begin{pmatrix} Y_{L_n^{(\kappa)}}^{(\kappa)} \\ \frac{Y_{L_n^{(\kappa)}}^{(\kappa)}}{Y^{(\kappa)}} \\ \tau_{L_n^{(\kappa)}}^{(\kappa)}(Y^{(\kappa)}) \end{pmatrix}.$$

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- Sequential simulation only over a very short time interval $[0, L_n^{(\kappa)}]$
- Most of the “activity” takes place in the first few sticks, where simulation is additive and hence stable!

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- use bridges of Y and $Y^{(\kappa)}$ to couple $(Y, Y^{(\kappa)})$ and use SBA coupling to construct $(X, X^{(\kappa)})$

$$\mathcal{W}_p((X_t, \overline{X}_t), (X_t^{(\kappa)}, \overline{X}_t^{(\kappa)})) \text{ \& } \mathcal{W}_1(\tau_t(X), \tau_t(X^{(\kappa)}))$$

Theorem

$\exists$ universal constants $K_1 := 1/2, K_2 > 0$, such that for any $\kappa \in (0, 1]$ and $t > 0$, there exists a coupling $(X, X^{(\kappa)})$ satisfying

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$$\mu_2(\kappa, t) := \min \left\{ \sqrt{2t} \bar{\sigma}_\kappa, \frac{K_2 \kappa \bar{\sigma}_\kappa}{(\bar{\sigma}_\kappa^2 + \sigma^2)^{1/2}} \right\} \left(1 + \log^+ \left(\frac{\sqrt{2t} (\bar{\sigma}_\kappa^2 + \sigma^2)^{1/2}}{K_2 \kappa} \right) \right),$$

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- Theorem holds for **all** Lévy processes
- analogous bounds for any $p \in [1, 2]$ hold
- dependence on time horizon t is explicit

Coupling the marginals X_t and $X_t^{(\kappa)}$

Mariucci and Reiß [MR18]: $\exists$ universal constants K_p depending only on $p \in [1, 2]$ such that for any $t > 0$ and $\kappa \in (0, 1]$ there exists a coupling between X_t and $X_t^{(\kappa)}$ satisfying

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Coupling the marginals X_t and $X_t^{(\kappa)}$

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$$\mathbb{E}[|X_t - X_t^{(\kappa)}|^p]^{1/p} \leq \min\{\sqrt{2t\bar{\sigma}_\kappa}, K_p \kappa\}.$$

For $p \in [1, 2]$, we establish the following bound

$$\mathbb{E}[|X_t - X_t^{(\kappa)}|^p]^{1/p} \leq \min\left\{\sqrt{2t\bar{\sigma}_\kappa}, \frac{K_{\lceil p \rceil} \kappa \bar{\sigma}_\kappa^{2/\lceil p \rceil}}{(\bar{\sigma}_\kappa^2 + \sigma^2)^{1/\lceil p \rceil}}\right\},$$

and apply it with the SBA coupling to get the upper bound

Coupling the marginals X_t and $X_t^{(\kappa)}$

Mariucci and Reiß [MR18]: $\exists$ universal constants K_p depending only on $p \in [1, 2]$ such that for any $t > 0$ and $\kappa \in (0, 1]$ there exists a coupling between X_t and $X_t^{(\kappa)}$ satisfying

$$\mathbb{E}[|X_t - X_t^{(\kappa)}|^p]^{1/p} \leq \min\{\sqrt{2t\bar{\sigma}_\kappa}, K_p \kappa\}.$$

For $p \in [1, 2]$, we establish the following bound

$$\mathbb{E}[|X_t - X_t^{(\kappa)}|^p]^{1/p} \leq \min\left\{\sqrt{2t\bar{\sigma}_\kappa}, \frac{K_{\lceil p \rceil} \kappa \bar{\sigma}_\kappa^{2/\lceil p \rceil}}{(\bar{\sigma}_\kappa^2 + \sigma^2)^{1/\lceil p \rceil}}\right\},$$

and apply it with the SBA coupling to get the upper bound

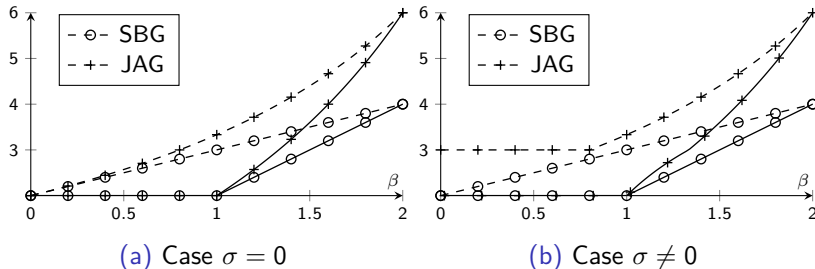
$$\mu_p(\kappa, t) = \min\left\{\sqrt{2t\bar{\sigma}_\kappa}, \frac{K_{\lceil p \rceil} \kappa \bar{\sigma}_\kappa^{2/\lceil p \rceil}}{(\bar{\sigma}_\kappa^2 + \sigma^2)^{1/\lceil p \rceil}}\right\} \left(1 + \log^+\left(\frac{\sqrt{2t}(\bar{\sigma}_\kappa^2 + \sigma^2)^{1/\lceil p \rceil}}{K_{\lceil p \rceil} \kappa \bar{\sigma}_\kappa^{2/\lceil p \rceil - 1}}\right)\right)$$

Coupling: from $(X, X^{(\kappa)})$ to $(X^{(\kappa)}, X^{(\kappa')})$

The coupling $(X, X^{(\kappa)})$ is *synchronous*, except for $(\bar{\sigma}_\kappa W, J^{1,\kappa})$ the increments of which cannot be samples.

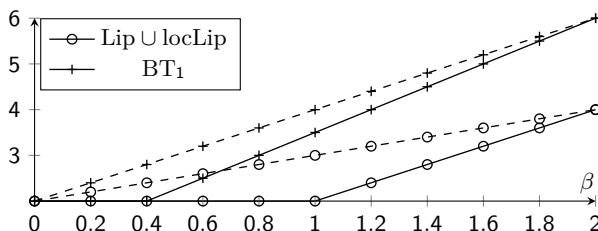
This same strategy can be used to couple $(X^{(\kappa)}, \bar{X}_t^{(\kappa)}, \tau_t(X^{(\kappa)}))$ & $(X^{(\kappa')}, \bar{X}_t^{(\kappa')}, \tau_t(X^{(\kappa')}))$, where $0 < \kappa < \kappa'$, with access to the sampling of increments. This coupling is crucial for the implementation of MLMC algorithms. In particular, we have the following consequences.

MC and MLMC complexity: comparison with JAG



The dominant power of ϵ^{-1} in the computational complexity as a function of the BG index β : estimation of $\mathbb{E}[f(X_T, \bar{X}_T)]$, where f is globally Lipschitz. Dashed (resp. solid) lines represent the MC (resp. MLMC) cost. The jump-adapted Gaussian approximation (JAG) plots are based on [Der11, DH11].

MC and MLMC complexity

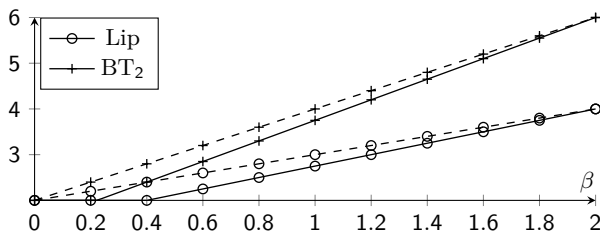


The picture shows dominant power of ϵ^{-1} in the computational complexity, as a function of the BG index β , of estimating $\mathbb{E}[f(X_T, \bar{X}_T)]$ where

$$|f(x, y) - f(x', y')| \leq K(|x - x'| + |y - y'|)e^{\lambda y \vee y'}, \quad (f \in \text{Lip} \cup \text{locLip}) \quad \text{or} \\ f(x, y) = g(x)\mathbb{1}_{\{y > y_0\}}, \quad (f \in \text{BT}_1).$$

Dashed (resp. solid) lines represent the MC (resp. MLMC) cost.

MC and MLMC complexity for $\mathbb{E}[f(X_T, \bar{X}_T, \tau_T)]$

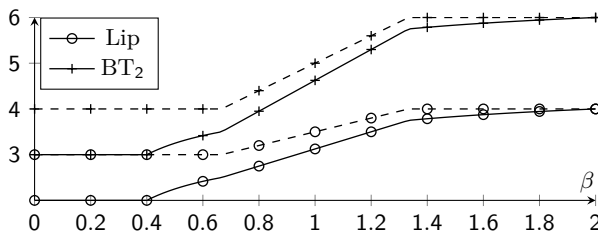


Assuming $\sigma \neq 0$, the picture shows dominant power of ϵ^{-1} in the computational complexity, as a function of the BG index β , of estimating $\mathbb{E}[f(X_T, \bar{X}_T, \tau_T)]$ where

$$|f(x, y, t) - f(x', y', t')| \leq K(|x - x'| + |y - y'| + |t - t'|), \quad (f \in \text{Lip}) \quad \text{or} \\ f(x, y, t) = g(x, y) \mathbb{1}_{\{t > t_0\}}, \quad \text{where } g \in \text{Lip}, \quad (f \in \text{BT}_2).$$

Dashed (resp. solid) lines represent the MC (resp. MLMC) cost.

MC and MLMC complexity for $\mathbb{E}[f(X_T, \bar{X}_T, \tau_T)]$



Assuming $\bar{\sigma}_\kappa^2 \sim c\kappa^{2-\beta}$ as $\kappa \rightarrow 0$, the picture shows dominant power of ϵ^{-1} in the computational complexity, as a function of the BG index β , of estimating $\mathbb{E}[f(X_T, \bar{X}_T, \tau_T)]$ where

$$|f(x, y, t) - f(x', y', t')| \leq K(|x - x'| + |y - y'| + |t - t'|), \quad (f \in \text{Lip}) \quad \text{or} \\ f(x, y, t) = g(x, y)\mathbb{1}_{\{t > t_0\}}, \quad \text{where } g \in \text{Lip}, \quad (f \in \text{BT}_2).$$

Dashed (resp. solid) lines represent the MC (resp. MLMC) cost.

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